Vector Calculus

exercice 5 :

$$\begin{cases} x = \rho\cos\theta \\ y = \rho\sin\theta \end{cases}$$

$$f(x,y) = f(\rho\cos\theta,\ \rho\sin\theta)$$

$$= F(\rho,\theta)$$

- $$\frac{\partial f}{\partial x} = \frac{\partial F}{\partial \rho} \times \frac{\partial \rho}{\partial x} + \frac{\partial F}{\partial \theta} \times \frac{\partial \theta}{\partial x}$$

- $$\frac{\partial f}{\partial y} = \frac{\partial F}{\partial \rho} \times \frac{\partial \rho}{\partial y} + \frac{\partial F}{\partial \theta} \times \frac{\partial \theta}{\partial y}.$$

on a $$\boxed{\begin{cases} \rho = \sqrt{x^2+y^2} \\ \theta = \arctan\left(\frac{y}{x}\right) \end{cases}}$$

d'où :

$$x \frac{\partial f}{\partial x}(x,y) + y \frac{\partial f}{\partial y}(x,y) = \sqrt{x^2+y^2}$$

devient.

$$x\left(\frac{\partial F}{\partial \rho} \times \frac{\partial \rho}{\partial x} + \frac{\partial F}{\partial \theta} \times \frac{\partial \theta}{\partial x}\right) + y\left(\frac{\partial F}{\partial \rho} \times \frac{\partial \rho}{\partial y} + \frac{\partial F}{\partial \theta} \times \frac{\partial \theta}{\partial y}\right) = \rho$$

or $\frac{\partial \rho}{\partial x} = \frac{x}{\rho}$ $\qquad$ $\frac{\partial \rho}{\partial y} = \frac{y}{\rho}$

$$\frac{\partial \theta}{\partial x} = \frac{-\frac{y}{x^2}}{1+\left(\frac{y}{x}\right)^2} = \frac{-\frac{y}{\cancel{x^2}}}{\frac{x^2+y^2}{\cancel{x^2}}} = \frac{-y}{\rho^2}$$

$$\frac{\partial \theta}{\partial y} = \frac{\frac{1}{x}}{1+\left(\frac{y}{x}\right)^2} = \frac{\frac{1}{x}}{\frac{x^2+y^2}{x^2}} = \frac{x}{\rho^2}$$

et alors :

$$x\left(\frac{\partial F}{\partial \rho}\times\frac{x}{\rho} + \frac{\partial F}{\partial \theta}\times\frac{-y}{\rho^2}\right)$$

$$+\, y\left(\frac{\partial F}{\partial \rho}\times\frac{y}{\rho} + \frac{\partial F}{\partial \theta}\times\frac{x}{\rho^2}\right) = \rho$$

$$\frac{\partial F}{\partial \rho}\times\frac{x^2}{\rho} + \cancel{\frac{\partial F}{\partial \theta}\times\frac{-xy}{\rho^2}}$$

$$+\, \frac{\partial F}{\partial \rho}\times\frac{y^2}{\rho} + \cancel{\frac{\partial F}{\partial \theta}\times\frac{xy}{\rho^2}} = \rho$$

$$\frac{\partial F}{\partial e} \times \frac{x^2}{e} + \frac{\partial F}{\partial e} \times \frac{y^2}{e} = e$$

$$\frac{\partial F}{\partial e} \times \frac{x^2 + y^2}{e} = e$$

$$\frac{\partial F}{\partial e} \times \frac{e^2}{e} = e$$

$$\frac{\partial F}{\partial e} \times e = e$$

$$\frac{\partial F}{\partial e} = 1.$$

par intégration par rapport à e:

$$\int \frac{\partial F}{\partial e}\, de = \int 1\, de$$

$$F(e,\theta) = e + G(\theta)$$

D'où :

$$f(x,y) = \sqrt{x^2+y^2} + G\left(\arctan\left(\frac{y}{x}\right)\right)$$

exercice 6 :

$$f(x,y) = f\left(\frac{u+v}{2} ; \frac{u-v}{2}\right)$$

$$= F(u,v).$$

$$\cdot \frac{\partial f}{\partial x} = \frac{\partial F}{\partial u} \times \frac{\partial u}{\partial x} + \frac{\partial F}{\partial v} \times \frac{\partial v}{\partial x}$$

$$= \frac{\partial F}{\partial u} + \frac{\partial F}{\partial v}$$

$$\cdot \frac{\partial^2 f}{\partial x^2}(x,y) = \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial u}(u,v) + \frac{\partial F}{\partial v}(u,v)\right)$$

or $$\frac{\partial}{\partial x}\left(\frac{\partial F}{\partial u}\right) = \frac{\partial^2 F}{\partial^2 u^2} \times \overbrace{\frac{\partial u}{\partial x}}^{=1} + \frac{\partial^2 F}{\partial v \partial u} \times \overbrace{\frac{\partial v}{\partial x}}^{=1}$$

$$\frac{\partial}{\partial x}\left(\frac{\partial F}{\partial u}\right) = \frac{\partial^2 F}{\partial u^2} + \frac{\partial^2 F}{\partial v \partial u}.$$

et

$$\frac{\partial}{\partial x}\left(\frac{\partial F}{\partial v}\right) = \frac{\partial^2 F}{\partial u \partial v} \times \underbrace{\frac{\partial u}{\partial x}}_{=1} + \frac{\partial^2 F}{\partial v^2} \times \underbrace{\frac{\partial v}{\partial x}}_{=1}.$$

$$= \frac{\partial^2 F}{\partial u \partial v} + \frac{\partial^2 F}{\partial v^2}.$$

donc :

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial^2 F}{\partial u^2} + 2\frac{\partial^2 F}{\partial u \partial v} + \frac{\partial^2 F}{\partial v^2}$$

- $$\frac{\partial f}{\partial y} = \frac{\partial F}{\partial u} \times \frac{\partial u}{\partial y} + \frac{\partial F}{\partial v} \times \frac{\partial v}{\partial y}$$

$$= \frac{\partial F}{\partial u} \times c + \frac{\partial F}{\partial v} \times -c$$

$$= c\left(\frac{\partial F}{\partial u} - \frac{\partial F}{\partial v}\right)$$

or :

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y} \times c\left(\frac{\partial F}{\partial u} - \frac{\partial F}{\partial v}\right)$$

$$= c\left[\frac{\partial}{\partial y}\frac{\partial F}{\partial u} - \frac{\partial}{\partial y}\frac{\partial F}{\partial v}\right]$$

cc : $\frac{\partial}{\partial y}\left(\frac{\partial F}{\partial u}\right)$

$$= \frac{\partial^2 F}{\partial u^2} \times \frac{\partial u}{\partial y} + \frac{\partial^2 F}{\partial v \partial u} \times \frac{\partial v}{\partial y}$$

$$= \frac{\partial^2 F}{\partial u^2} \times c + \frac{\partial^2 F}{\partial v \partial u} \times - c$$

$$\frac{\partial}{\partial y}\left(\frac{\partial F}{\partial v}\right) = \frac{\partial^2 F}{\partial u \partial v} \times \frac{\partial u}{\partial y}$$

$$+ \frac{\partial^2 F}{\partial v^2} \times \frac{\partial v}{\partial y}.$$

$$= \frac{\partial^2 F}{\partial u \partial v} \times c + \frac{\partial^2 F}{\partial v^2} \times - c$$

donc :

$$\frac{\partial^2 f}{\partial y^2} = c\left[\frac{\partial^2 F}{\partial u^2}\times c + \frac{\partial^2 F}{\partial v \partial u}\times -c - \left(\frac{\partial^2 F}{\partial u \partial v}\times c + \frac{\partial^2 F}{\partial v^2}\times -c\right)\right]$$

$$= c^2\left[\frac{\partial^2 F}{\partial u^2} + \frac{\partial^2 F}{\partial v^2} - 2\frac{\partial^2 F}{\partial u \partial v}\right]$$

Donc :

$$\frac{\partial^2 f}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 f}{\partial y^2} = 0$$

$$\cancel{\frac{\partial^2 F}{\partial u^2}} + 2\frac{\partial^2 F}{\partial u \partial v} + \cancel{\frac{\partial^2 F}{\partial v^2}} - \cancel{\frac{\partial^2 F}{\partial u^2}} - \cancel{\frac{\partial^2 F}{\partial v^2}} + 2\frac{\partial^2 F}{\partial u \partial v} = 0$$

$$\Leftrightarrow \frac{\partial^2 F}{\partial u \partial v} = 0$$

$$\frac{\partial^2 F}{\partial u \partial v} = 0.$$

donc $F(u,v) = F_1(u) + F_2(v)$

et d'où :

$$f(x,y) = F_1(x+cy) + F_2(x-cy).$$