Integral

execice 1 :

$f(x,y) = xy$ continuous.

1) $I = \iint_{\mathcal{U}} xy \, dx\,dy.$

$\mathcal{U} = \{ (x,y) \in \mathbb{R}^2, \ x \geq 0, y \geq 0, \ x + y \leq 1 \}$

$\Rightarrow 0 \leq y \leq 1 - x$ $\geq 0 \Rightarrow x \leq 1$

Fubini

$$I = \int_{x=0}^{x=1} \int_{y=0}^{y=1-x} xy \, dy\,dx$$

$$= \int_{x=0}^{x=1} x \left[\frac{y^2}{2} \right]_0^{1-x} dx$$

$$= \int_{x=0}^{x=1} x \frac{(1-x)^2}{2} dx$$

$$= \frac{1}{2} \int_{x=0}^{x=1} x(1-2x+x^2) \, dx$$

$$= \frac{1}{2} \int_{x=0}^{x=1} x - 2x^2 + x^3 \, dx$$

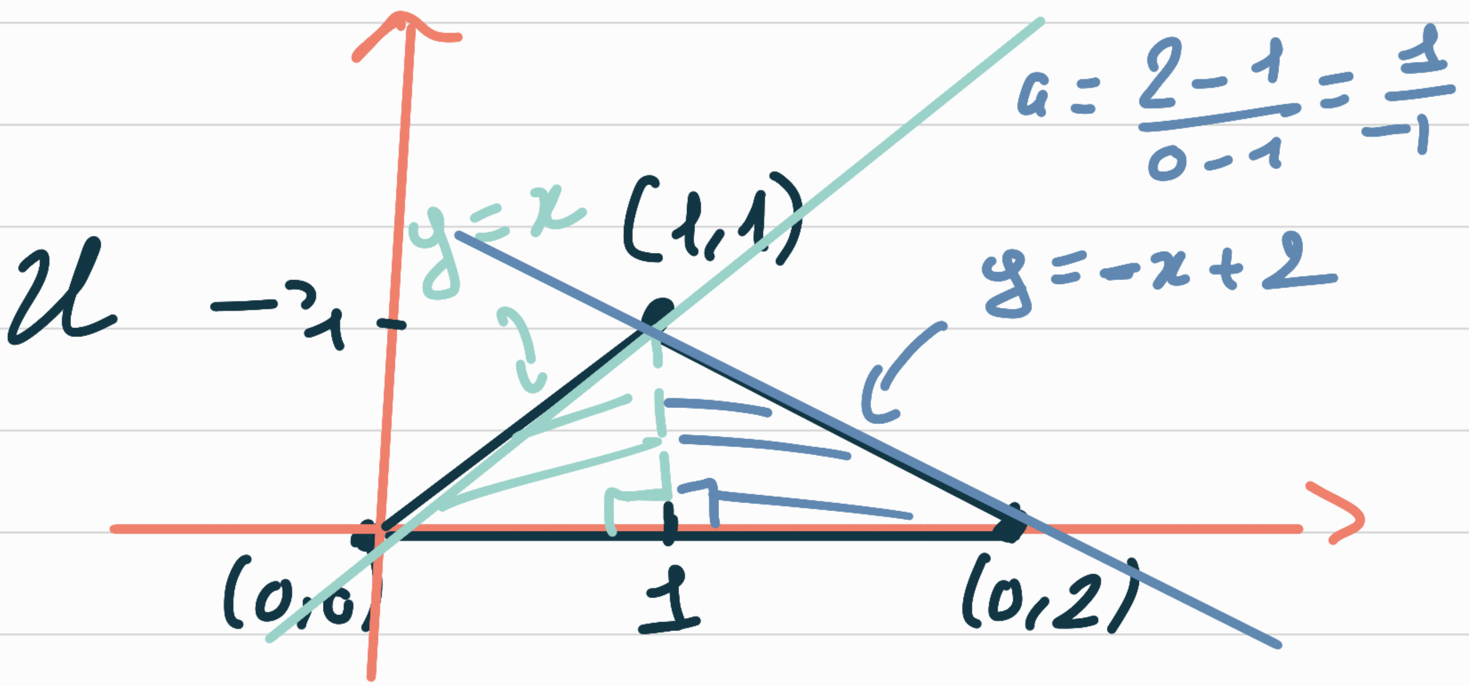
$$= \frac{1}{2} \left[\frac{x^2}{2} - \frac{2}{3} x^3 + \frac{x^4}{4} \right]_0^1$$

$$= \frac{1}{2}\left(\underbrace{\frac{1}{2} - \frac{2}{3} + \frac{1}{4}}_{\frac{3}{4} - \frac{2}{3} = \frac{9-8}{12} = \frac{1}{12}}\right)$$

$$= \frac{1}{2} \times \frac{1}{12}$$

$$= \frac{1}{24}.$$

2) $$I = \iint_{U} x + y \; dx\,dy.$$



- $$\begin{cases} 0 \le x \le 1 , & 0 \le y \le x \\ 1 \le x \le 2 , & 0 \le y \le -x+2 \end{cases}$$

$$I = \underbrace{\int_{x=0}^{x=1} \int_{y=0}^{y=x} x+y \, dy \, dx}_{I_1} + \underbrace{\int_{x=1}^{x=2} \int_{y=0}^{y=-x+2} x+y \, dy \, dx}_{I_2}$$

$$\bullet \; I_1 = \int_{x=0}^{x=1} \int_{y=0}^{y=x} x + y \, dy \, dx$$

$$= \int_{x=0}^{x=1} \left[xy + \frac{y^2}{2} \right]_0^x dx$$

$$= \int_{x=0}^{x=1} x^2 + \frac{x^2}{2} \, dx$$

$$= \int_{x=0}^{x=1} \frac{3}{2} x^2 \, dx$$

$$= \frac{3}{2} \left[\frac{x^3}{3} \right]_0^1 = \frac{\cancel{3}}{2} \times \frac{1}{\cancel{3}} = \frac{1}{2}$$

$$\bullet \quad I_2 = \int_{x=1}^{x=2} \int_{y=0}^{y=-x+2} x + y \; dy\, dx$$

$$= \int_{x=1}^{x=2} \left[xy + \frac{y^2}{2} \right]_0^{-x+2} dx$$

$$= \int_{x=1}^{x=2} x(-x+2) + \frac{(-x+2)^2}{2} \, dx$$

$$= \int_{x=1}^{x=2} -x^2 + 2x + \frac{4 - 4x + x^2}{2} \, dx$$

$$= \int_{x=1}^{x=2} -x^2 + 2x + 2 - 2x + \frac{x^2}{2} \, dx$$

$$= \int_{x=1}^{x=2} -\frac{1}{2}x^2 + 2 \, dx$$

$$= \left[-\frac{1}{2} \times \frac{1}{3}x^3 + 2x \right]_1^2$$

$$= \left(-\frac{1}{2} \times \frac{8}{3} + 4 \right) - \left(-\frac{1}{2} \times \frac{1}{3} + 2 \right)$$

$$= \left(-\frac{4}{3} + 4 \right) - \left(-\frac{1}{6} + 2 \right)$$

$$= \frac{8}{3} - \frac{11}{6}$$

$$= \frac{5}{6}.$$

donc $I = I_1 + I_2$

$$= \frac{1}{2} + \frac{5}{6}$$

$$= \frac{8}{6}$$

$$= \frac{4}{3}.$$

exercice 4 :

1) • intersection points

$$x^2 + y^2 = 36 - 3x^2 - 3y^2$$

$$4x^2 + 4y^2 = 36$$

$$x^2 + y^2 = 9$$

$$\Rightarrow \mathcal{D}((0,0), 3) = \{ x^2 + y^2 \leq 9 \}$$

• Coordonnées cylindriques :

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases}, \quad \begin{array}{l} \theta \in [0; 2\pi[\\ r \geq 0 \end{array}$$

$\to z = x^2 + y^2$

$= (r\cos\theta)^2 + (r\sin\theta)^2$

$= r^2(\cos^2\theta + \sin^2\theta)$

$= r^2 \to$ bottom

$\to z = 36 - 3x^2 - 3y^2$

$= 36 - 3r^2 . \longrightarrow$ top.

$\Rightarrow 36 - 3r^2 \geq r^2$

$\Leftrightarrow 36 \geq 4r^2$

$\Leftrightarrow 9 \geq r^2$

• Calcul de V

$$V = \iint_{D((0,0),3)} Z_{top} - Z_{bottom} \, dV$$

$$= \int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=3} \left[\underbrace{36 - 3r^2 - r^2}_{36 - 4r^2} \right] r \, dr \, d\theta$$

$$= \left(\int_{\theta=0}^{\theta=2\pi} 1 \, d\theta \right) \left(\int_{r=0}^{r=3} 36r - 4r^3 \, dr \right)$$

$$= 2\pi \times \left[\frac{\cancel{36}^{\,18} r^2}{\cancel{2}} - \frac{\cancel{4} r^4}{\cancel{4}} \right]_0^3$$

$$= 2\pi\left(18 \times 9 - 81\right)$$

$$= 2\pi \times 81$$

$$= 162\pi.$$

polar / cylindre : $r\,dr\,d\theta$

sphere : $r^2 \sin(\phi)\,dr\,d\theta\,d\phi$.