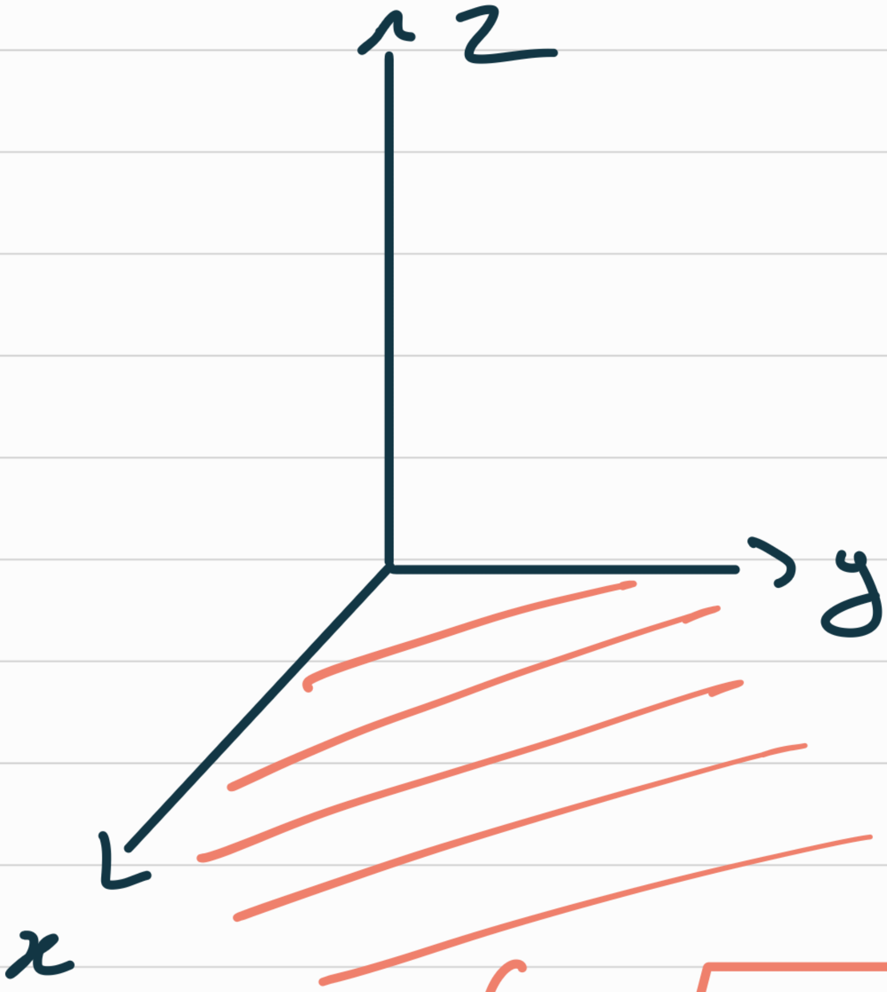
Integrals

exercice 3 :

2)



$\Rightarrow \boxed{z = 0}$

On passe en coordonnées cylindrique :

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases}, \quad \begin{array}{l} r \geq 0 \\ \theta \in [0;2\pi[\end{array}$$

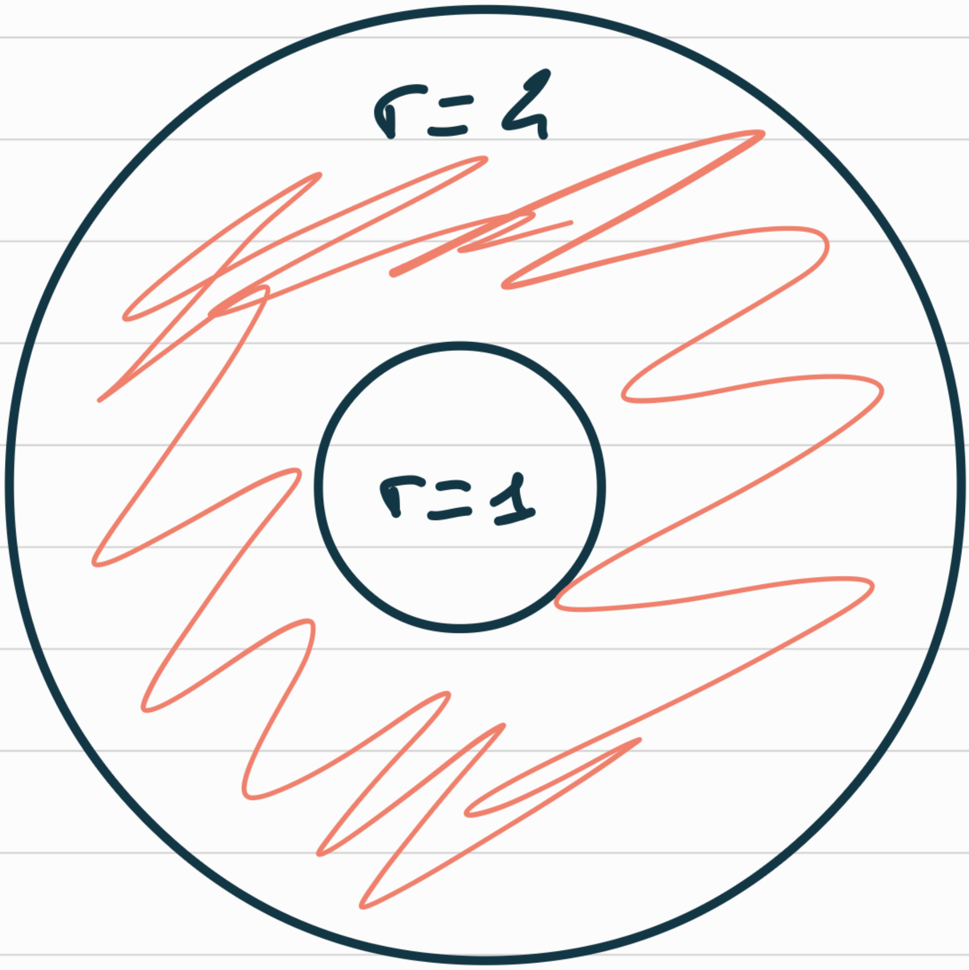
- $x^2 + y^2 = 1 \iff r^2 = 1$

  $\iff r = 1$

- $x^2 + y^2 = 4 \iff r^2 = 4$

  $\iff r = 2$

$$\Rightarrow \boxed{1 \leq r \leq 2}$$



- $dx\,dy\,dz = r\,dr\,d\theta\,dz$
- $0 \le z \le x + 2$

$$\boxed{0 \le z \le r\cos\theta + 2}$$

$$\bullet \iiint_E z \, dx\,dy\,dz$$

$$= \int_{\theta=0}^{\theta=2\pi} \int_{r=1}^{r=2} \int_{z=0}^{z=r\cos\theta+2} z \; r \, dz\, dr\, d\theta$$

$$= \int_{\theta=0}^{\theta=2\pi} \int_{r=1}^{r=2} r \left[\frac{z^2}{2} \right]_0^{r\cos\theta+2} dr\, d\theta$$

$$= \int_{\theta=0}^{\theta=2\pi} \int_{r=1}^{r=2} r \, \frac{(r\cos\theta+2)^2}{2} \, dr\, d\theta$$

$$= \int_{\theta=0}^{\theta=2\pi} \int_{r=1}^{r=2} r \frac{(r\cos\theta+2)^2}{2} \, dr \, d\theta$$

$$= \int_{\theta=0}^{\theta=2\pi} \int_{r=1}^{r=2} \frac{r^3\cos^2\theta + 4r^2\cos\theta + 4r}{2} \, dr \, d\theta$$

$$= \int_{\theta=0}^{\theta=2\pi} \int_{r=1}^{r=2} \frac{r^3\cos^2\theta}{2} + 2r^2\cos\theta + 2r \, dr \, d\theta$$

$$= \int_{\theta=0}^{\theta=2\pi} \left[\frac{r^4\cos^2\theta}{8} + \frac{2}{3} r^3\cos\theta + r^2 \right]_1^2 d\theta$$

$$= \int_{\theta=0}^{\theta=2\pi} \left[\frac{16\cos^2\theta}{8} + \frac{2}{3} \times 8\cos\theta + 4 - \frac{\cos^2\theta}{8} - \frac{2}{3}\cos\theta - 1 \right] d\theta$$

$$= \int_{\theta=0}^{\theta=2\pi} \frac{15}{8}\cos^2\theta + \frac{14}{3}\cos\theta + 3 \, d\theta$$

$$= \int_{\theta=0}^{\theta=2\pi} \frac{15}{8}\cos^2\theta \, d\theta + \int_{\theta=0}^{\theta=2\pi} \frac{14}{3}\cos\theta \, d\theta$$

$$+ 3 \int_{\theta=0}^{\theta=2\pi} 1 \, d\theta$$

$$\left[\frac{14}{3}\sin\theta \right]_0^{2\pi} = 0$$

$$= 3 \times 2\pi = 6\pi$$

$$\cdot \frac{15}{8} \int_{\theta=0}^{\theta=2\pi} \cos^2\theta \, d\theta$$

$$= \frac{15}{8} \int_{\theta=0}^{\theta=2\pi} \frac{1+\cos(2\theta)}{2} \, d\theta$$

$$= \frac{15}{16} \left[\theta + \underbrace{\frac{1}{2}\sin(2\theta)}_{=0} \right]_{0}^{2\pi}$$

$$= \frac{15}{16} \times 2\pi$$

$$= \frac{15}{8}\pi .$$

donc :

$$\iiint_E z \, dx\,dy\,dz$$

$$= \frac{15}{8}\pi + 0 + 6\pi$$

$$= \frac{63}{8}\pi$$

3) On passe en coordonnées sphérique :

$$\begin{cases} x = r\cos(\theta)\sin(\phi) \\ y = r\sin(\theta)\sin(\phi) \\ z = r\cos(\phi) \end{cases}$$

$r \geq 0$; $\theta \in [0 ; 2\pi[$; $\phi \in [0, \pi[$

$$dx\,dy\,dz = r^2 \sin(\phi)\, dr\, d\theta\, d\phi.$$

- $E = B^{\mathbb{R}^3}(0, 1) \Rightarrow 0 \leq r \leq 1$

$$x^2+y^2+z^2$$

$$= r^2\cos^2(\theta)\sin^2(\phi) + r^2\sin^2(\theta)\sin^2(\phi) + r^2\cos(\phi)$$

$$= r^2\sin^2(\phi)\big(\underbrace{\cos^2\theta + \sin^2\theta}_{=1}\big) + r^2\cos(\phi)$$

$$= r^2\sin^2\phi + r^2\cos^2\phi$$

$$= r^2\left(\sin^2\phi + \cos^2\phi\right)$$

$$= r^2$$

$$\iiint_E x^2+y^2+z^2 \, dx\,dy\,dz$$

$$= \int_0^{2\pi}\int_0^{\pi}\int_0^{1} r^2 \times r^2\sin(\phi)\, dr\,d\phi\,d\theta$$

$$= \int_0^{2\pi} \int_0^{\pi} \sin\phi \int_0^1 r^4 \, dr \, d\phi \, d\theta$$

$$= \left[-\cos(\phi)\right]_0^{\pi} = 1 + 1 = 2$$

$$= \underbrace{\left(\int_0^{2\pi} 1 \, d\theta\right)}_{2\pi} \left(\int_0^{\pi} \sin\phi \, d\phi\right) \underbrace{\left(\int_0^1 r^4 dr\right)}_{\left[\frac{r^5}{5}\right]_0^1 = \frac{1}{5}}$$

$$= 2\pi \times 2 \times \frac{1}{5}$$

$$= \frac{4\pi}{5}.$$

exercice 4 :

3)

$$\begin{cases} x = 2u \\ y = 3v \\ z = 5w \end{cases}$$

$$\frac{x^2}{4} + \frac{y^2}{9} + \frac{z^2}{25} = 1$$

$$\hookrightarrow u^2 + v^2 + w^2 = 1$$

$x(u,v,w)$

$y(u,v,w)$

$z(u,v,w)$

$$\det\begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{pmatrix}$$

$$= \begin{vmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{vmatrix}$$

$$= 2 \times 15$$

$$= 30 \implies dx\,dy\,dz = 30\, du\,dv\,dw$$

$$\mathrm{Vol}(\text{ellip}) = \iiint_{B^3(0,1)} 1 \times 30\, du\,dv\,dw$$

$$= 30 \times \mathrm{Vol}(B(0,1))$$

$$= 30 \times \frac{4\pi}{3} = 40\pi.$$