Worksheet 4

exercice 7 :

$$h(u,v) = f(uv, u^2+v^2)$$

$$= f\big(x(u,v), y(u,v)\big)$$

avec $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$.

$$h = f \circ g$$

où $g : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$

$$(u,v) \longmapsto (\underbrace{uv}_{x(u,v)}, \underbrace{u^2+v^2}_{y(x,y)})$$

donc, $h : \mathbb{R}^2 \longrightarrow \mathbb{R}$

$$\Rightarrow J^{(u,v)} h \in \mathcal{M}_{12}(\mathbb{R})$$

→ **Way 1**: Chain Rule

- $\frac{\partial h}{\partial u}(u,v) = \frac{\partial f}{\partial x} \times \frac{\partial x}{\partial u}$

$$+ \frac{\partial f}{\partial y} \times \frac{\partial y}{\partial u}$$

$$= \frac{\partial f}{\partial x} \times v$$

$$+ \frac{\partial f}{\partial y} \times 2u$$

- $\frac{\partial h}{\partial v}(u,v) = \frac{\partial f}{\partial x} \times \frac{\partial x}{\partial v}$

$$+ \frac{\partial f}{\partial y} \times \frac{\partial y}{\partial v}$$

$$= \frac{\partial f}{\partial x} \times u + \frac{\partial f}{\partial y} \times 2v$$

donc avec $x(u,v) = uv \quad y(u,v) = u^2 + v^2$

$$J^{(u,v)} h = \begin{pmatrix} v \frac{\partial f}{\partial x} + 2u \frac{\partial f}{\partial y} & u \frac{\partial f}{\partial x} + 2v \frac{\partial f}{\partial y} \end{pmatrix}$$

→ Way 2 : Matrix Chain Rule

$$h = f \circ g \Rightarrow J^{(u,v)} h = J^{g(u,v)} f \times J^{(u,v)} g$$

$(uv, u^2+v^2) = (x,y)$

$$J^{(x,y)} f = \begin{pmatrix} \frac{\partial f}{\partial x}(x,y) & \frac{\partial f}{\partial y}(x,y) \end{pmatrix}$$

$$J^{(u,v)} g = \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix}$$

$$= \begin{pmatrix} v & u \\ 2u & 2v \end{pmatrix}$$

$$J^{(u,v)} h = \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \end{pmatrix} \begin{pmatrix} v & u \\ 2u & 2v \end{pmatrix}$$

$1 \times \cancel{2} \times \cancel{2} \times 2$

$$= \begin{pmatrix} \frac{\partial f}{\partial x}(x,y) \times v + \frac{\partial f}{\partial y}(x,y) \times 2u & \frac{\partial f}{\partial x}(x,y) \times u + \frac{\partial f}{\partial y}(x,y) \times 2v \end{pmatrix}$$

Worksheet 6.

exercice 5 :

2) on pose $\begin{cases} u = xy \\ v = y^2 - x^2 \end{cases}$

$u(x,y)$ et $v(x,y) \Rightarrow du\,dv = \left|J^{(x,y)}_{(u,v)}\right| dx\,dy$

$x(u,v)$ et $y(u,v) \Rightarrow dx\,dy = \left|J^{(u,v)}_{(x,y)}\right| du\,dv$

$$J^{(x,y)}_{(u,v)} = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix}$$

$$= \begin{pmatrix} y & x \\ -2x & 2y \end{pmatrix}$$

$$\left| J^{(x,y)}_{(u,v)} \right| = y \times 2y - (-2x \times x)$$

$$= 2y^2 + 2x^2$$

$$= 2(y^2 + x^2).$$

donc $\quad du\,dv = 2(y^2 + x^2)\,dx\,dy$

$$\frac{du\,dv}{2} = (y^2 + x^2)\,dx\,dy.$$

- $0 < x < y \Rightarrow u = xy > 0$
- $0 < x < y \Rightarrow x^2 < y^2$

$$\Rightarrow 0 < y^2 - x^2$$

$$\Rightarrow v > 0$$

$0 < a < b$

- $a < xy < b \Rightarrow a < u < b$
- $y^2 - x^2 < 1 \Rightarrow v < 1.$

Donc :

$$\iint_{\mathcal{U}} (y^2 - x^2)^{xy} \underbrace{(x^2+y^2)\, dx\, dy}_{\frac{du\, dv}{2}}$$

$$= \frac{1}{2} \int_a^b \int_0^1 v^u \, dv\, du$$

$$= \frac{1}{2} \int_a^b \left[\frac{v^{u+1}}{u+1} \right]_0^1 du$$

$$= \frac{1}{2} \int_a^b \frac{1}{u+1} \, du$$

$$= \frac{1}{2}\left[\ln(u+1) \right]_a^3$$

$$= \frac{1}{2}\left(\ln(3+1) - \ln(a+1) \right)$$

$$= \frac{1}{2}\ln\left(\frac{3+1}{a+1}\right).$$