Line Integral

- ## Fundamental theorem

$$\int_a^b F'(x)\,dx = F(b) - F(a)$$

$$\int_{\mathcal{C}} \vec{\nabla} f \cdot d\vec{r} = f(\underbrace{r(b)}_{\text{fin } \mathcal{C}}) - f(\underbrace{r(a)}_{\text{début } \mathcal{C}})$$

$$\underbrace{\mathcal{C} = r(t) \quad , \quad a \le t \le b}_{\text{paramétrage}}$$

$$\int_{C_1} \vec{\nabla} f \cdot d\vec{r} = \int_{C_2} \vec{\nabla} f \cdot d\vec{r}$$

exercice :

1. trouver la fonction potentielle associée au champ de vecteurs

2. $$W = \int_C \vec{F} \cdot d\vec{r}$$

$$= \phi(B) - \phi(A)$$

1) $\vec{F} = \nabla \phi$

$$\vec{F}(x, y, z) = m M g \begin{pmatrix} \dfrac{-x}{(x^2+y^2+z^2)^{3/2}} \\ -y/\cdot \\ -z/\cdot \end{pmatrix}$$

on pose $r = \sqrt{x^2+y^2+z^2}$

$$= -mMg \begin{pmatrix} \dfrac{x}{r^3} \\ y/r^3 \\ z/r^3 \end{pmatrix}$$

or $\nabla\left(\frac{1}{r}\right) = \left(-\frac{x}{r^3}, \frac{-y}{r^3}, -\frac{z}{r^3}\right)$

$$\frac{\partial}{\partial x}\left(\frac{1}{\sqrt{x^2+y^2+z^2}}\right)$$

$$= -\frac{1}{\cancel{2}} \times \cancel{2}x \times (x^2+y^2+z^2)^{-\frac{1}{2}-1}$$

$$F = mMg \times \nabla\left(\frac{1}{r}\right)$$

$$= \nabla\left(\frac{mMg}{r}\right)$$

donc $\phi(r) = \frac{mMg}{r}$.

② $W = \int_C \vec{F} \cdot d\vec{r}$

$= \Phi(2,2,0) - \Phi(3,4,12)$

$= \dfrac{m\pi g}{\sqrt{2^2+2^2+0}} - \dfrac{m\pi g}{\sqrt{3^2+4^2+12^2}}$

$= \dfrac{m\pi g}{2\sqrt{2}} - \dfrac{m\pi g}{13}$

$= m\pi g \left(\dfrac{13 - 2\sqrt{2}}{26\sqrt{2}} \right)$

exercice :

1) on pose $P(x,y) = 3 + 2xy$

$$Q(x,y) = x^2 - 3y^2$$

$$\frac{\partial P}{\partial y}(x,y) = 2x$$

$$\frac{\partial Q}{\partial x}(x,y) = 2x$$

donc $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ sur

$\mathbb{R}^2$ (simply connected)

et F est conservative.

2) Soit f telle que

$F = \nabla f$

$$\begin{cases} \dfrac{\partial f}{\partial x} = 3 + 2xy \\ \dfrac{\partial f}{\partial y} = x^2 - 3y^2 \end{cases}$$

donc $f(x,y) = 3x + x^2 y + g(y)$

or $\dfrac{\partial f}{\partial y} = \cancel{x^2} + g'(y) = \cancel{x^2} - 3y^2$

$\Rightarrow g'(y) = -3y^2$ et $g(y) = -y^3 + c$

donc : $f(x,y) = 3x + x^2 y - y^3 + C, \quad C \in \mathbb{R}$

3) $\int_C F \cdot dr = f(r(\pi)) - f(r(0))$

$= f(0, -e^{\pi}) - f(0, 1)$

$= -(-e^{\pi})^3 - (-(1)^3)$

$= \boxed{e^{3\pi} + 1}$

$\partial D = \mathcal{C}$

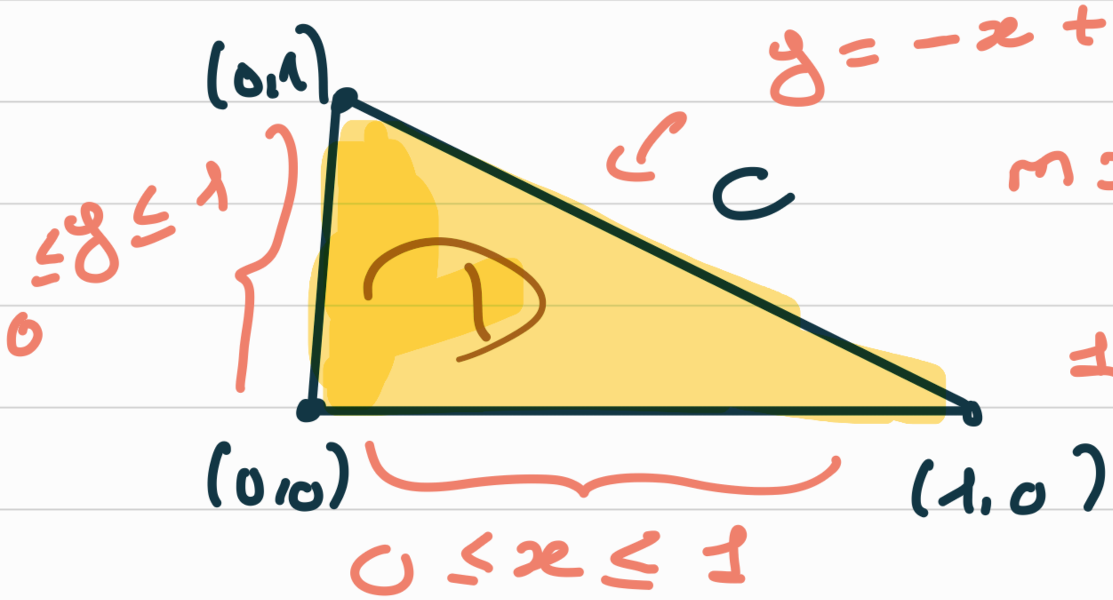
↳ bord

exercice :

$$\oint_C x^4\,dx + xy\,dy$$

$P(x,y) = x^4 \qquad Q(x,y) = xy$

$$\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$



$y = -x + 1$

$m = \dfrac{1-0}{0-1} = -1$

$1 = -0 + p$

$$\frac{\partial P}{\partial y} = 0 \qquad \frac{\partial Q}{\partial x} = y.$$

$$\oint_C x^4\,dx + xy\,dy = \iint_D y\,dA.$$

$$= \int_0^1 \int_0^{1-x} y\,dy\,dx$$

$$= \int_0^1 \left[\frac{y^2}{2}\right]_0^{1-x}$$

$$= \int_0^1 \frac{(1-x)^2}{2}\,dx$$

$$= \frac{1}{2}\int_0^1 1-2x+x^2 \, dx$$

$$= \frac{1}{2}\left[x - x^2 + \frac{x^3}{3}\right]_0^1$$

$$= \frac{1}{2} \times \left(1 - 1 + \frac{1}{3}\right)$$

$$= \frac{1}{6} .$$