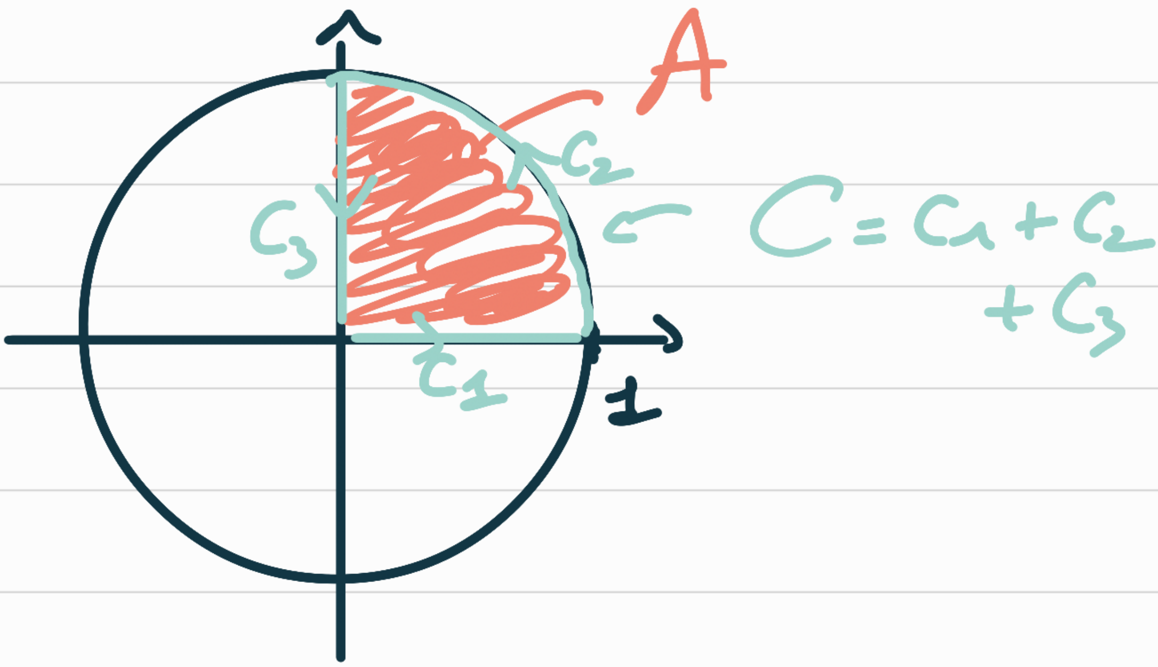
Line Integral

exercice 5 :

①



- $x^2 + y^2 \leq 1 \iff x^2 \leq 1 - y^2$

$$\iff 0 \leq x \leq \sqrt{1-y^2}$$

- $0 \leq y \leq 1$

$$I = \iint_{\mathcal{U}} x + y \; dx\,dy.$$

$$= \int_0^1 \int_0^{\sqrt{1-y^2}} (x+y)\,dx\,dy$$

$$= \int_0^1 \left[\frac{x^2}{2} + xy \right]_0^{\sqrt{1-y^2}} dy$$

$u = 1-y^2$

$u' = -2y$

$u'u^n \Rightarrow \frac{u^{n+1}}{n+1}$

$\Rightarrow y(1-y^2)^{\frac{1}{2}}$

$$= \int_0^1 \frac{1}{2} - \frac{y^2}{2} + y\sqrt{1-y^2}\; dy$$

$$= \left[\frac{1}{2}y - \frac{y^3}{6} - \frac{1}{2} \times \frac{(1-y^2)^{\frac{3}{2}}}{\frac{3}{2}} \right]_0^1$$

$$= \left[\frac{1}{2}y - \frac{y^3}{6} - \frac{1}{3}(1-y^2)^{3/2}\right]_0^1$$

$$= \frac{1}{2} - \frac{1}{6} - \frac{1}{3} \times 0 - \left(-\frac{1}{3} \times 1\right)$$

$$= \frac{1}{2} - \frac{1}{6} + \frac{1}{3}$$

$$= \frac{5}{6} - \frac{1}{6}$$

$$= \frac{4}{6}$$

$$= \frac{2}{3}.$$

② $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$; $0 \leq r \leq 1$, $0 \leq \theta \leq \frac{\pi}{2}$

$$x + y = r\cos\theta + r\sin\theta$$

$$= r(\cos\theta + \sin\theta).$$

$$dx\,dy = r\,dr\,d\theta$$

$$I = \iint_{\mathcal{U}} x + y \; dx\,dy$$

$$= \int_0^{\pi/2}\int_0^{1} r(\cos\theta + \sin\theta) \cdot r \; dr\,d\theta$$

$$= \int_0^{\pi/2} \int_0^1 r^2 (\cos\theta + \sin\theta)\, dr\, d\theta$$

$$= \left[\int_0^{\pi/2} (\cos\theta + \sin\theta)\, d\theta \right] \left[\int_0^1 r^2\, dr \right]$$

$$= \left[\sin\theta - \cos\theta \right]_0^{\pi/2} \times \left[\frac{r^3}{3} \right]_0^1$$

$$= (1 + 1) \times \frac{1}{3}$$

$$= \frac{2}{3}$$

3 $$\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

$$= \oint_C P\,dx + Q\,dy.$$

On cherche (P, Q) tel que :

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = x + y.$$

on pose $\begin{cases} Q(x,y) = \frac{x^2}{2} \\ P(x,y) = -\frac{y^2}{2} \end{cases}$

et on vérifie bien que :

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = x + y \quad \checkmark$$

Donc d'après la formule de Green :

$$I = \iint_{\mathcal{U}} x + y \, dx dy$$

$$= \oint_{C} P dx + Q dy .$$

avec : $\begin{cases} Q(x,y) = \dfrac{x^2}{2} \\ P(x,y) = -\dfrac{y^2}{2} \end{cases}$

C_1 : x-axis $(0,0) \to (1,0)$

C_2 : quarter circle from $(1,0)$ to $(0,1)$

C_3 : y-axis $(0,1) \to (0,0)$.

- $C_1 \Rightarrow (\underset{x}{t}, \underset{y}{0})$, $t \in [0,1]$

$dx = dt$ et $dy = 0$

$P = -\frac{y^2}{2} = 0 \qquad Q = \frac{t^2}{2}$

$$\int_{C_1} P\,dx + Q\,dy = \int_0^1 0 \cdot dt + \frac{t^2}{2} \times 0$$

$$= 0$$

- $C_3 \Rightarrow (\underset{x}{0}, \underset{y}{t}) \quad t \in [0;1]$

$dx = 0 \qquad dy = dt$

$P = -\frac{t^2}{2} \qquad Q = 0$

$$\oint_{C_3} P\,dx + Q\,dy$$

$$= \int_1^0 -\frac{t^2}{2} \times 0 + 0 \times dt = 0$$

$r = 1$

- $C_2 \Rightarrow (r\cos\theta, r\sin\theta)$, $\theta \in [0; \frac{\pi}{2}]$

$\underbrace{r\cos\theta}_{x}$, $\underbrace{r\sin\theta}_{y}$

$dx = -\sin\theta\, d\theta \qquad dy = \cos\theta\, d\theta$

$P = -\dfrac{\sin^2\theta}{2} \qquad Q = \dfrac{\cos^2\theta}{2}$.

$$\oint_{C_2} P\,dx + Q\,dy$$

$$= \int_0^{\pi/2} -\frac{\sin^2\theta}{2} \times -\sin\theta\, d\theta + \frac{\cos^2\theta}{2} \times \cos\theta\, d\theta$$

$$= \frac{1}{2}\int_0^{\pi/2} \sin^3\theta + \cos^3\theta \, d\theta.$$

$$\text{or} \int_0^{\pi/2} \sin^3\theta \, d\theta = \frac{2}{3}$$

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$$\int_0^{\pi/2} \cos^3\theta \, d\theta = \frac{2}{3}$$

$$\text{donc} \quad I = \frac{1}{2}\left(\frac{2}{3} + \frac{2}{3}\right)$$

$$= \frac{2}{3}$$