Line integral

exercice 4 :

$$I = \oint_{\Gamma} \frac{(x-y)\,dx + (x+y)\,dy}{x^2+y^2}$$

1. $\Gamma = \{(x,y) \in \mathbb{R}^2 : x^2+y^2 = 1\}$

on pose : $P(x,y) = \dfrac{x-y}{x^2+y^2}$

$$Q(x,y) = \frac{x+y}{x^2+y^2}$$

d'où $I = \oint_{\Gamma} P\,dx + Q\,dy$.

On paramétrise Γ :

$$\begin{cases} x = \cos(\theta) \\ y = \sin(\theta) \end{cases}, \quad \theta \in [0; 2\pi].$$

$(x-y) = \cos(\theta) - \sin(\theta)$

$(x+y) = \cos\theta + \sin(\theta)$

$x^2 + y^2 = 1$

$dx = -\sin(\theta)\,d\theta$

$dy = \cos(\theta)\,d\theta$

$$\cdot (x-y)\,dx + (x+y)\,dy$$

$$= \big(\cos(\theta) - \sin(\theta)\big) \times -\sin(\theta)\,d\theta$$
$$+ \big(\cos(\theta) + \sin(\theta)\big) \times \cos(\theta)\,d\theta$$

$$= \left[-\cancel{\cos\theta \sin\theta} + \sin^2\theta + \cos^2\theta + \cancel{\sin\theta\cos\theta} \right] d\theta$$

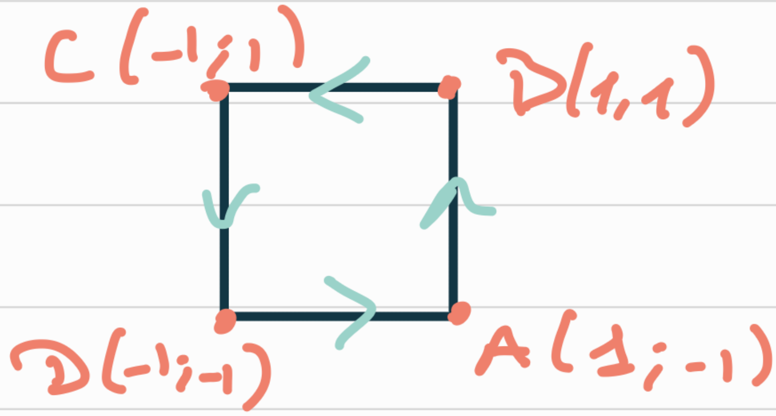
$$= d\theta \, .$$

$$\frac{(x-y)\,dx + (x+y)\,dy}{x^2+y^2} = \frac{1}{1}\,d\theta = d\theta$$

donc :

$$I = \int_0^{2\pi} 1 \, d\theta = 2\pi$$

② Γ :



on pose : $P(x,y) = \dfrac{x-y}{x^2+y^2}$

$Q(x,y) = \dfrac{x+y}{x^2+y^2}$

$$I = \oint_\Gamma P\,dx + Q\,dy$$

• AB : $x = 1$ et $y = t$

avec $t \in [-1;1]$.

$dx = 0$ et $dy = dt$

$$I_{AB} = \int_{-1}^{1} \frac{1-t}{1+t^2} \times 0 + \frac{1+t}{1+t^2}\, dt$$

$$= \int_{-1}^{1} \frac{1+t}{1+t^2}\, dt$$

$$= \int_{-1}^{1} \frac{1}{1+t^2} + \frac{t}{1+t^2}\, dt$$

$$= \left[\arctan(t) + \frac{1}{2} \ln(1+t^2) \right]_{-1}^{1}$$

$$= \frac{\pi}{4} + \frac{1}{2}\ln(2) - \left(-\frac{\pi}{4} + \frac{1}{2}\ln(2)\right)$$

$$= \frac{\pi}{2}.$$

- BC : $y = 1$ et $x = t$

$t \in [1 ; -1]$

$dy = 0$ et $dx = dt$

$$I_{BC} = \int_{1}^{-1} \frac{t-1}{1+t^2}\, dt$$

$$= - \int_{-1}^{1} \frac{-(1-t)}{1+t^2}\, dt$$

$$= + \int_{-1}^{1} \frac{1}{1+t^2} + \frac{t}{1+t^2} \, dt$$

$$= \left[\arctan(t) + \frac{1}{2} \ln(1+t^2) \right]_{-1}^{1}$$

$$= \frac{\pi}{2}$$

- CD : $x = -1$ $y = t$

$t \in [1 ; -1]$

$dx = 0$ $dy = t$

$$I_{CD} = \int_{1}^{-1} \frac{-1 + t}{1+t^2} \, dt$$

$$= -\int_{-1}^{1} \frac{t-1}{1+t^2}\,dt$$

$$= \frac{\pi}{2}.$$

- $\underline{DA}$: $y = -1$ $\quad x = t$

$t \in [-1;1]$

$$I_{DA} = \int_{-1}^{1} \frac{t+1}{1+t^2}\,dt$$

$$= \frac{\pi}{2}$$

Donc $I = 4 \times \frac{\pi}{2} = 2\pi.$