Fiche 6

exercice 3 :

1) Soit $x \in \mathbb{R}$.

On pose $\sum_{n \geq 0} u_n$, $u_n = \dfrac{x^{3n}}{(3n)!}$

$$\left| \frac{u_{n+1}}{u_n} \right| = \left| \frac{\dfrac{x^{3n+1}}{(3n+3)!}}{\dfrac{x^{3n}}{(3n)!}} \right|$$

$$= \left| \frac{x^{\cancel{3n}+3}}{(3n+3)!} \times \frac{(3n)!}{\cancel{x^{3n}}} \right|$$

$$= \left| x^3 \times \frac{\cancel{(3n)!}}{(3n+3)(3n+2)(3n+1)\cancel{(3n)!}} \right|$$

$$= \frac{|x^3|}{(3n+3)(3n+2)(3n+1)} \xrightarrow[n \to +\infty]{} 0 < 1.$$

donc la série $\sum u_n$ CVA par tout $x \in \mathbb{R}$.

Donc le rdc est $+\infty$, et ainsi $\forall x \in \mathbb{R}$ f est C^∞ et en particulier $f^{(3)}$ existe.

De plus,

$$f^{(3)}(x) = \sum_{n \geq 1} \frac{3n(3n-1)(3n-2)\, x^{3n-3}}{(3n)!}$$

pour que $3n-3 \geq 0$

$3n - 3 \geq 0 \iff n \geq 1.$

2) $$f(x) = \sum_{n \geq 0} \frac{1}{(3n)!} x^{3n}$$

et $$f^{(3)}(x) = \sum_{n \geq 1} \frac{3n(3n-1)(3n-2)}{(3n)!} x^{3n-3}$$

or $$\frac{3n(3n-1)(3n-2)}{(3n)!} = \frac{\cancel{3n}\cancel{(3n-1)}\cancel{(3n-2)}}{\cancel{3n}\cancel{(3n-1)}\cancel{(3n-2)}(3n-3)!}$$

$$= \frac{1}{(3n-3)!}$$

d'où $$f^{(3)}(x) = \sum_{n \geq 1} \frac{1}{(3n-3)!} x^{3n-3}$$

$m = n-1$

$$= \sum_{m \geq 0} \frac{1}{(3m)!} x^{3m}$$

$$= f(x).$$

Donc pour tout $x \in \mathbb{R}$:

$$f^{(3)}(x) = f(x)$$

$$\iff f^{(3)}(x) - f(x) = 0.$$

E_c $r^3 - 1 = 0$

d'où $r^3 = 1$.

$-\frac{1}{2} - i\frac{\sqrt{3}}{2}$

Les solutions : $\{1 ; j ; j^2\}$

les conjugués

avec $j = e^{\frac{2i\pi}{3}} = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$

α β

remarque : $j^2 = \left(e^{2i\pi/3}\right)^2 = e^{\frac{4i\pi}{3}}$

$$j^3 = \left(e^{\frac{2i\pi}{3}}\right)^3 = e^{2i\pi} = 1$$

$$(j^2)^3 = j^6 = (j^3)^2 = 1.$$

donc :

$$f(x) = C_1 e^{x} + C_2 e^{-\frac{1}{2}x} \cos\left(\frac{\sqrt{3}}{2}x\right)$$
$$+ C_3 e^{-\frac{1}{2}x} \sin\left(\frac{\sqrt{3}}{2}x\right).$$

or $$f(0) = \underbrace{1}_{n=0} + \sum_{n \geq 1} \frac{0^{3n}}{(3n)!} = 1$$

$$f'(0) = 0$$

$\mathcal{E}$, $$f'(x) = \sum_{n \geq 1} 3n \frac{x^{3n-1}}{(3n)!} = \sum_{n \geq 1} \frac{x^{3n-1}}{(3n-1)!}$$

$$f''(x) = \sum_{n \geq 1} \frac{3n-1\, x^{3n-2}}{(3n-1)!}$$

$$= \sum_{n \geq 1} \frac{x^{3n-2}}{(3n-2)!}$$

donc $f''(0) = 0$.

$$f(0) = C_1 + C_2$$

$$f'(x) = C_1 e^{x} - \frac{1}{2} C_2 e^{-\frac{1}{2}x} \cos\left(\frac{\sqrt{3}}{2}x\right) - \frac{\sqrt{3}}{2} C_2 e^{-\frac{1}{2}x} \sin\left(\frac{\sqrt{3}}{2}x\right) - \frac{1}{2} C_3 e^{-\frac{1}{2}x} \sin\left(\frac{\sqrt{3}}{2}x\right) + \frac{\sqrt{3}}{2} C_3 e^{-\frac{1}{2}x} \cos\left(\frac{\sqrt{3}}{2}x\right)$$

$$f'(0) = C_1 - \frac{1}{2} C_2 + \frac{\sqrt{3}}{2} C_3.$$

$$f''(x) = C_1 e^{x} + \frac{1}{4} C_2 e^{-\frac{1}{2}x} \cos\left(\frac{\sqrt{3}}{2}x\right) + \frac{1}{2} \times \frac{\sqrt{3}}{2} C_2 e^{-\frac{1}{2}x} \sin\left(\frac{\sqrt{3}}{2}x\right) + \frac{\sqrt{3}}{4} C_2 e^{-\frac{1}{2}x} \sin\left(\frac{\sqrt{3}}{2}x\right)$$

$$-\frac{3}{4} C_2 e^{-\frac{1}{2}x} \cos\left(\frac{\sqrt{3}}{2}x\right) + \frac{1}{4} C_3 e^{-\frac{1}{2}x} \sin\left(\frac{\sqrt{3}}{2}x\right)$$

$$-\frac{\sqrt{3}}{4} C_3 e^{-\frac{1}{2}x} \cos\left(\frac{\sqrt{3}}{2}x\right) - \frac{\sqrt{3}}{4} C_3 e^{-\frac{1}{2}x} \cos\left(\frac{\sqrt{3}}{2}x\right)$$

$$-\frac{3}{4} C_3 e^{-\frac{1}{2}x} \sin\left(\frac{\sqrt{3}}{2}x\right).$$

e)

$$y''(0) = C_1 + \frac{1}{4} C_2 - \frac{3}{4} C_2 - \frac{\sqrt{3}}{4} C_3 - \frac{\sqrt{3}}{4} C_3$$

$$= C_1 - \frac{1}{2} C_2 - \frac{\sqrt{3}}{2} C_3 .$$

Donc :

$$\begin{cases} C_1 + C_2 = 1 \\ C_1 - \frac{1}{2} C_2 + \frac{\sqrt{3}}{2} C_3 = 0 \\ C_1 - \frac{1}{2} C_2 - \frac{\sqrt{3}}{2} C_3 = 0 \end{cases}$$

$$\iff c_1 = c_2 = c_3 = \frac{1}{3}.$$

Donc :

$$f(x) = \frac{1}{3}\left(e^x + \underbrace{e^{jx} + e^{j^2x}}\right)$$

$$\cos\left(\frac{2\pi}{3}\right) + i\cancel{\sin\left(\frac{2\pi}{3}\right)} + \cos\left(\frac{2\pi}{3}\right) - i\cancel{\sin\left(\frac{2\pi}{3}\right)}$$

$$= 2\cos\left(\frac{2\pi}{3}\right)$$

$$= \frac{1}{3}\left(e^x + 2e^{-\frac{1}{2}x}\cos\left(\frac{\sqrt{3}}{2}x\right)\right)$$

3) $\displaystyle\sum_{n\geq 0} \frac{1}{(3n)!} = f(1) = \frac{1}{3}\left(e + 2e^{-\frac{1}{2}}\cos\left(\frac{\sqrt{3}}{2}\right)\right)$