Fiche 5 :

exercice 6 :

$$y'' - 3y' + 2y = \frac{1}{1+e^{-2t}} .$$

$$y_h(t) = C_2\, e^{t} + C_1 e^{2t} .$$

on pose $y_p(t) = C_2(t)\, e^{t} + C_1(t) e^{2t}$.

avec $\boxed{C_2'(t)\, e^{t} + C_1'(t) e^{2t} = 0}$

$$y_p'(t) = C_2'(t) e^{t} + C_2(t) e^{t} + C_1'(t) e^{2t} + 2C_1(t) e^{2t}$$

$$= C_2(t) e^{t} + 2C_1(t) e^{2t} .$$

$$y_p''(t) = C_2'(t)e^t + C_2(t)e^t$$
$$+ \overset{1}{\cancel{2}}C_1'(t)e^{2t} + 4C_1(t)e^{2t}.$$
$$= C_2(t)e^t + C_1'(t)e^{2t} + 4C_1(t)e^{2t}$$

$$y_p'' - 3y_p' + 2y_p$$
$$= \cancel{C_2(t)e^t} + C_1'(t)e^{2t} + \cancel{4C_1(t)e^{2t}}$$
$$- \cancel{3C_2(t)e^t} - \cancel{6C_1(t)}e^{2t} + \cancel{2C_2e^t}$$
$$+ \cancel{2C_1e^{2t}}$$

$$= C_1'(t)e^{2t}.$$

Fiche 6

exercice 1 :

1) $(1-t^2)y'' - 2ty' + 2y = 0.$

on pose $y_p(t) = t$

$(1-t^2) \times 0 - 2t \times 1 + 2t = 0.$

donc $y_p(t) = t$ est une solution particulière.

$(1-t^2)y'' - 2ty' + 2y = 0$

$\Leftrightarrow y'' - \frac{2t}{(1-t^2)}y' + \frac{2}{1-t^2}y = 0$

Abaissement de l'ordre :

① cherche une solution particulière y_p.

② on pose $y(t) = z(t) \times y_p(t)$

③ on dérive et on remplace dans l'équation, et on simplifie

④ on pose $x(t) = z'(t)$, et on résoud EDO 1.

⑤ on remonte à $z(t)$ puis $y(t)$.

on pose $y(t) = z(t) \times y_p(t)$.

$$= t\,z(t).$$

$$y'(t) = z(t) + t\,z'(t)$$

$$y''(t) = z'(t) + t\,z''(t) + z'(t)$$

$$= t\,z''(t) + 2z'(t).$$

$$(1-t^2)y'' - 2t y' + 2y = 0.$$

$$\iff (1-t^2)\big(t z''(t) + 2z'(t)\big) - 2t\big(z(t) + t z'(t)\big) + 2t z(t) = 0$$

$$\iff t(1-t^2)z''(t) + 2(1-t^2)z'(t) - \cancel{2t z(t)} - 2t^2 z'(t) + \cancel{2t z(t)} = 0$$

$$\iff t(1-t^2)z''(t) + \big[\underbrace{2(1-t^2) - 2t^2}_{2-2t^2-2t^2 = 2-4t^2}\big]z'(t) = 0$$

$$\iff t(1-t^2)z''(t) + 2(1-2t^2)z'(t) = 0$$

on pose $x(t) = z'(t)$.

$$t(1-t^2)x'(t) + 2(1-2t^2)x(t) = 0$$

$$\iff \underbrace{x'(t) + \frac{2(1-2t^2)}{t(1-t^2)}x(t) = 0}_{\text{EDL 1.}}$$

$$\Rightarrow x(t) = C e^{-\int \frac{2(1-2t^2)}{t(1-t^2)}}, \quad \underline{\underline{C \in \mathbb{R}}}$$

$$\hookrightarrow -2\int \frac{(1-2t^2)}{t(1-t^2)}\,dt \quad ?$$

$$\frac{1-2t^2}{t(1-t^2)} = \frac{1}{t(1-t^2)} - \frac{2t^{\cancel{2}}}{\cancel{t}(1-t^2)}$$

$$= \underbrace{\frac{1}{t(1-t^2)}} - \underbrace{\frac{2t}{(1-t^2)}}_{\ln(1-t^2)}$$

$$\frac{1}{t(1-t)(1+t)} = \frac{a}{t} + \frac{b}{1-t} + \frac{c}{1+t}$$

$$a = \lim_{t \to 0} \frac{1}{(1-t)(1+t)} = 1$$

$$b = \lim_{t \to 1} \frac{1}{t(1+t)} = \frac{1}{2}.$$

$$c = \lim_{t \to -1} \frac{1}{t(1-t)} = -\frac{1}{2}$$

$$\frac{1}{t(1-t^2)} = \frac{1}{t} + \frac{1}{2(1-t)} - \frac{1}{2(1+t)}$$

$$\int \frac{1}{t(1-t^2)} = \ln(t) - \frac{1}{2}\ln(1-t) - \frac{1}{2}\ln(1+t)$$

Donc :

$$-2\int \frac{(1-2t^2)}{t(1-t^2)}\,dt$$

$$= -2\left[\ln(t) - \frac{1}{2}\ln(1-t) - \frac{1}{2}\ln(1+t) + \ln(1-t^2)\right]$$

$$= -2\ln(t) + \overbrace{\ln(1-t) + \ln(1+t)}^{=\ln((1-t)(1+t)) = \ln(1-t^2)}$$
$$- 2\ln(1-t^2)$$

$$= -2\ln(t) + \ln(1-t^2) - 2\ln(1-t^2)$$

$$= -2\ln(t) - \ln(1-t^2)$$

Donc :

$$-\int \frac{2(1-2t^2)}{t(1-t^2)}\,dt = -2\ln(t) - \ln(1-t^2)$$

$$= \ln(t^{-2}) + \ln\left((1-t^2)^{-1}\right)$$

$$= \ln\left(t^{-2}(1-t^2)^{-1}\right)$$

D'où :

$$x(t) = C\,e^{\ln\left(t^{-2}(1-t^2)^{-1}\right)}$$

$$= \frac{C}{t^2(1-t^2)}$$

or $x(t) = z'(t)$

donc $z(t) = \int \frac{C}{t^2(1-t^2)}$

$$z(t) = \frac{c}{2}\left(-\frac{2}{t} - \ln(1-t) + \ln(1+t)\right)$$

et finalement :

$$y(t) = t\,z(t)$$

$$= \frac{ct}{2}\left(-\frac{2}{t} - \ln(1-t) + \ln(1+t)\right)$$