EDO

exercice 3 :

1) $$y' = ay + b$$

$$\hookrightarrow \boxed{y(t) = k e^{at} - \frac{b}{a}, \quad k \in \mathbb{R}}$$

$$y(t) = k e^{t} - 1, \quad k \in \mathbb{R}.$$

2) $\Rightarrow$ on suppose f solution de (E)

donc $$f' - \left(1 + \frac{\cos x}{\sin x}\right) f = \sin x.$$

or $f(x) = g(x)\sin(x)$.

et $f'(x) = g'(x)\sin(x) + g(x)\cos(x)$.

d'où.

$$g'(x)\sin(x) + g(x)\cos(x) - \left(1 + \frac{\cos(x)}{\sin(x)}\right)\times g(x)\sin(x) = \sin(x)$$

$$g'(x)\sin(x) + g(x)\cos(x) - g(x)\sin(x) - \frac{\cos(x)}{\cancel{\sin(x)}} \times g(x)\cancel{\sin(x)} = \sin(x).$$

$$g'(x)\sin(x) + \cancel{g(x)\cos(x)} - g(x)\sin(x) - \cancel{\cos(x)\,g(x)} = \sin(x)$$

$g'(x)\sin(x) - g(x)\sin(x) = \sin(x)$

or $x \in]0; \frac{\pi}{2}[$, donc $\sin(x) \neq 0$

$\Rightarrow g'(x) - g(x) = 1$

donc g est solution de (E_0).

$\Leftarrow$) On suppose g solution de (E_0)

donc $g' - g = 1$

or $f(x) = g(x)\sin(x)$, d'où

$$g(x) = \frac{f(x)}{\sin(x)}, \quad \forall x \in]0; \tfrac{\pi}{2}[.$$

et $$g'(x) = \frac{f'(x)\sin(x) - f(x)\cos(x)}{\sin^2(x)}$$

on a alors :

$$\frac{f'(x)\sin(x) - f(x)\cos(x)}{\sin^2(x)} - \frac{f(x)}{\sin(x)} = 1$$

$$\frac{f'(x)\sin(x) - f(x)\cos(x) - \sin(x)f(x)}{\sin^2(x)} = 1$$

$$f'(x)\sin(x) - f(x)\cos(x) - \sin(x)f(x) = \sin^2(x)$$

$$\cancel{\sin(x)}\left[f'(x) - f(x)\frac{\cos(x)}{\sin(x)} - f(x)\right] = \sin^{\cancel{2}}(x)$$

$$f'(x) - f(x)\left(1 + \frac{\cos(x)}{\sin(x)}\right) = \sin(x)$$

donc f est solution de (E).

Donc . . .

3) f est solution de

(E) $\iff$ g est solution de (E_0).

$g(x) = ke^{x} - 1, \; k \in \mathbb{R}.$

d'où $f(x) = g(x)\sin(x)$

$= (ke^{x} - 1)\sin(x), \; k \in \mathbb{R}$

$= k\sin(x)e^{x} - \sin(x), \; k \in \mathbb{R}$

or $f\left(\frac{\pi}{4}\right) = k\sin\left(\frac{\pi}{4}\right)e^{\pi/4} - \sin\left(\frac{\pi}{4}\right)$

$= k\frac{\sqrt{2}}{2}e^{\pi/4} - \frac{\sqrt{2}}{2} = 0$

$$k = \frac{\frac{\sqrt{2}}{2}}{\frac{\sqrt{2}}{2} e^{\pi/4}} = e^{-\pi/4}$$

donc l'unique solution de (E) telle que $f\left(\frac{\pi}{4}\right) = 0$ est :

$$\boxed{f(x) = e^{-\frac{\pi}{4}} \sin(x)\, e^{x} - \sin(x).}$$

exercice 4 :

1) on suppose p solution de (E_2).

$$p' = \frac{1}{250} p \times (120 - p)$$

or $p(t) = \frac{1}{y(t)}$, et donc

$$p'(t) = -\frac{y'(t)}{y^2(t)}.$$

$$\frac{-y'(t)}{y^2(t)} = \frac{1}{250} \times \frac{1}{y(t)} \times \left(120 - \frac{1}{y(t)}\right)$$

$$\frac{-y'(t)}{y^2(t)} \times y^2 = \frac{120}{250} \times \frac{1}{y(t)} \times y^2 - \frac{1}{250} \times \frac{1}{y^2(t)} \times y^2$$

$$-y'(t) = 0,48\,y(t) - \frac{1}{250}$$

$$-y'(t) - 0,48\,y(t) = -\frac{1}{250} \quad \times -1$$

$$y'(t) + 0,48\,y(t) = \frac{1}{250}.$$

donc y est solution de (E_1).

2) • y est solution de (E_1)

$\Leftrightarrow$ p est solution de (E_2).

$$\Rightarrow \quad y' + 0,48y = \frac{1}{250}$$

$$\hookrightarrow \quad g(x) = k\,e^{-0,48x} + \frac{1}{120}, \quad k \in \mathbb{R}$$

a $p(t) = \dfrac{1}{y(t)}$

$$= \frac{1}{k e^{-0,48t} + \frac{1}{120}}$$

$$= \frac{1}{\frac{120k e^{-0,48t} + 1}{120}}$$

$$= \frac{120}{1 + \underbrace{120k}_{K} e^{-0,48t}}$$

on pose $K = 120k \in \mathbb{R}$.

et $p(t) = \dfrac{120}{1 + Ke^{-0,48t}}$, $K \in \mathbb{R}$

3) $p(0) = 30$

$$p(0) = \frac{120}{1+ke^0} = \frac{120}{1+k}$$

$$\Rightarrow \frac{120}{1+k} = 30$$

$$\Leftrightarrow 120 = 30(1+k)$$

$$\Leftrightarrow 120 = 30 + 30k$$

$$\Leftrightarrow 90 = 30k$$

$$\Leftrightarrow k = 3$$

donc l'unique solution

$$p(t) = \frac{120}{1 + 3e^{-0{,}48t}}.$$