TD 3

exercice 5 :

2) • $\lim_{h \to 0} \dfrac{f(0+h, 0) - f(0,0)}{h} = 0$

donc f admet une dérivée partielle p/r à x et $\dfrac{\partial f}{\partial x}(0,0) = 0$

• $\lim_{h \to 0} \dfrac{f(0, 0+h) - f(0,0)}{h}$

$= \lim_{h \to 0} \dfrac{3h^3/h^2 - 0}{h}$

$= \lim_{h \to 0} \dfrac{3\cancel{h^3}}{\cancel{h^3}} = 3$

donc f admet une dérivée partielle p/r à y et $\frac{\partial f}{\partial y}(0,0) = 3$.

3) $\frac{\partial f}{\partial x}(x,y) = \frac{-4xy^3}{(x^2+y^2)^2}$

$\frac{\partial f}{\partial y}(x,y) = \frac{x^4 + 8x^2y^2 + 3y^4}{(x^2+y^2)^2}$

4) $\lim_{x \to 0} \frac{\partial f}{\partial x}(x,0) = 0$

$\lim_{x \to 0} \frac{\partial f}{\partial x}(x,x) = \lim_{x \to 0} \frac{-4x^4}{4x^4}$

$= -1$

donc $\frac{\partial f}{\partial x}$ n'est pas continue en $(0,0)$ et f n'est pas de classe $\mathcal{C}^1$ sur $\mathbb{R}^2$.

5)

$$\boxed{z = f(x_0, y_0) + \frac{\partial f}{\partial x}(x_0, y_0) \times (x - x_0) + \frac{\partial f}{\partial y}(x_0, y_0) \times (y - y_0)}$$

- $f(1,1) = 2$
- $\frac{\partial f}{\partial x}(1,1) = -1$
- $\frac{\partial f}{\partial y}(1,1) = 3$

donc l'équation :

$$z = 2 - (x-1) + 3(y-1)$$

$$= 2 - x + 1 + 3y - 3$$

donc $\boxed{z = -x + 3y}$

$$\nabla f = \left(\frac{\partial f}{\partial x} \; ; \; \frac{\partial f}{\partial y} \right)$$

gradient

exercice 7 :

a) $\dfrac{x^2-y^2}{x^2+y^2}$, $\forall (x,y) \neq (0,0)$.

$$|x^2-y^2| \leq x^2+y^2$$

$$\left|\frac{x^2-y^2}{x^2+y^2}\right| \leq \frac{x^2+y^2}{x^2+y^2} \leq 1.$$

$$\boxed{|x \pm y| \leq |x|+|y|}$$

b) $$|f(x,y)| = \left|xy \times \frac{x^2-y^2}{x^2+y^2}\right|$$

$$\leq |xy| \times \left|\frac{x^2-y^2}{x^2+y^2}\right|$$

$$\leq |xy|$$

$$\leq \frac{x^2+y^2}{2}$$

$$\boxed{x^2+y^2 \geq \frac{|xy|}{2}}$$

$$\xrightarrow[(x,y)\to(0,0)]{} 0$$

d) Si f est $\mathcal{C}^2$ sur $\mathcal{U}$,

alors $\dfrac{\partial^2 f}{\partial^2 xy} = \dfrac{\partial^2 f}{\partial^2 yx}$

$$\frac{\partial f}{\partial x} \to \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2}$$

$$\frac{\partial f}{\partial x} \to \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial f}{\partial y \partial x}$$

$$\frac{\partial f}{\partial y} \longrightarrow \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y}$$

$$\frac{\partial f}{\partial y} \longrightarrow \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2}$$

e)

$$\lim_{h \to 0} \frac{\frac{\partial f}{\partial x}(0+h, 0) - \frac{\partial f}{\partial x}(0,0)}{h} = \frac{\partial^2 f}{\partial x^2}$$

$$\lim_{h \to 0} \frac{\frac{\partial f}{\partial x}(0; h+0) - \frac{\partial f}{\partial x}(0,0)}{h} = \frac{\partial^2 f}{\partial y \partial x}$$

$$\lim_{h \to 0} \frac{\frac{\partial f}{\partial y}(0+h, 0) - \frac{\partial f}{\partial y}(0,0)}{h} = \frac{\partial^2 f}{\partial x \partial y}$$

$$\lim_{h \to 0} \frac{\frac{\partial f}{\partial y}(0, 0+h) - \frac{\partial f}{\partial y}(0,0)}{h} = \frac{\partial^2 f}{\partial y^2}.$$