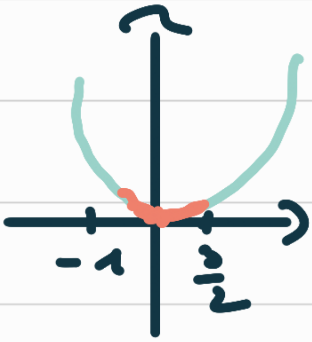
Probabilités

exercice 24 :

$$X \hookrightarrow \mathcal{U}\left(\left[\overset{Y=1}{-1}\,;\,\overset{Y=\frac{9}{4}}{\frac{3}{2}}\right]\right)$$

on pose $Y = X^2$.



- $Y(\Omega) = [0\,;\,\frac{9}{4}]$

- Soit $t \in \mathbb{R}$.

$$F_Y(t) = \mathbb{P}(Y \leq t)$$
$$= \mathbb{P}(X^2 \leq t)$$
$$= \mathbb{P}(-\sqrt{t} \leq X \leq \sqrt{t})$$

$$= \int_{-\sqrt{t}}^{\sqrt{t}} f_X(x)\,dx$$

$$= \int_{-\sqrt{t}}^{\sqrt{t}} \frac{1}{\frac{3}{2}-(-1)} \mathbb{1}_{[-1;\frac{3}{2}]}(x)\,dx$$

$$= \int_{-\sqrt{t}}^{\sqrt{t}} \frac{2}{5} \times \mathbb{1}_{[-1;\frac{3}{2}]}(x)\,dx$$

$\hookrightarrow$ $t \in [0;1]$

$[-\sqrt{t},\sqrt{t}] \subset [-1;\frac{3}{2}]$

$$= \left[\frac{2}{5}x\right]_{-\sqrt{t}}^{\sqrt{t}}$$

$$= \frac{2}{5}\sqrt{t} - \frac{2}{5}\times -\sqrt{t} = \frac{4}{5}\sqrt{t}$$

$t \in [0;1]$

$$\int_{-\sqrt{t}}^{\sqrt{t}} \frac{2}{5} \times \mathbb{1}_{[1\,;\,\frac{3}{2}]}(x)\, dx$$

$\hookrightarrow t \in [1\,;\frac{9}{4}]$

$\Rightarrow -\sqrt{t} > -\frac{3}{2}$

$$= \int_{-1}^{\sqrt{t}} \frac{2}{5}\, dx$$

$$= \left[\frac{2}{5}x\right]_{-1}^{\sqrt{t}}$$

$$= \frac{2}{5}\sqrt{t} - \frac{2}{5} \times -1$$

$$= \frac{2}{5}\sqrt{t} + \frac{2}{5}$$

Donc finalement :

$$F_Y(t) = \begin{cases} 0 & t < 0 \\ \frac{4}{5}\sqrt{t} & t \in [0,1] \\ \frac{2}{5}\sqrt{t} + \frac{2}{5} & t \in [1, \frac{9}{4}] \\ 1 & t > \frac{9}{4} \end{cases}$$

$$f_Y(t) = \begin{cases} \frac{2}{5\sqrt{t}} & t \in [0,1] \\ \frac{1}{5\sqrt{t}} & t \in [1; \frac{9}{4}] \\ 0 & \text{sinon} \end{cases}$$

exercice 26 :

1) $Y \hookrightarrow \mathcal{E}(\lambda)$. , $\lambda > 0$

$$E[e^Y] = \int_0^{+\infty} \underbrace{e^x \cdot \lambda e^{-\lambda x}}_{\exp(x)\cdot f_Y(x)} dx$$

$$= \int_0^{+\infty} \lambda e^{x(1-\lambda)} dx$$

. $1 - \lambda < 0 \iff \lambda > 1$.

$$E[e^Y] = \lambda \left[\frac{1}{1-\lambda} e^{x(1-\lambda)}\right]_0^{+\infty}$$

$$= 0 - \lambda \times \frac{1}{1-\lambda} e^0$$

$$= \frac{-\lambda}{1-\lambda} = \frac{\lambda}{\lambda-1}$$

- $1-\lambda \geq 0 \iff 0 < \lambda \leq 1$

$E[e^Y] = +\infty.$

Donc :

$$E[e^Y] = \begin{cases} \dfrac{\lambda}{\lambda-1} & \lambda > 1 \\ +\infty & 0 < \lambda \leq 1 \end{cases}$$

2) on pose $Z = \exp(Y)$.

. $Z(\Omega) = [1 ; +\infty[$.

. Soit $t \in \mathbb{R}$.

$t \geq 1$

$$F_Z(t) = \mathbb{P}(Z \leq t)$$

$$= \mathbb{P}(\exp(Y) \leq t)$$

$$= \mathbb{P}(Y \leq \ln(t))$$

$$= F_Y(\ln(t))$$

$$= \int_{-\infty}^{\ln(t)} f_Y(x)\, dx$$

$$= \int_0^{\ln(t)} \lambda e^{-\lambda x} \, dx$$

$$= \left[-e^{-\lambda x} \right]_0^{\ln(t)}$$

$$= -e^{-\lambda \ln(t)} - (-e^0)$$

$$= 1 - e^{\ln(t^{-\lambda})}$$

$$= 1 - t^{-\lambda} \quad , \quad t \geq 1$$

$$F_Z(t) = \begin{cases} 0 & t < 1 \\ 1 - t^{-\lambda} & t \geq 1 \end{cases}$$

$\cdot\ f_Z(t) = \lambda\, t^{-\lambda-1}\,\mathbb{1}_{t \geq 1}$

exercice 27 :

Soit f une fonction positive et mesurable.

$$E[f(X)] = E\left[f\left(-\frac{\ln(U)}{\lambda}\right)\right]$$

$X = -\frac{\ln(U)}{\lambda}$

$$= \int_{\mathbb{R}} f\left(\underbrace{-\frac{\ln(u)}{\lambda}}_{x}\right)\mathbb{1}_{]0;1[}(u)\,du$$

$$= \int_0^1 f\left(\frac{-\ln(u)}{\lambda}\right) du$$

on pose $x = -\frac{\ln(u)}{\lambda} \iff u = e^{-\lambda x}$

$$dx = -\frac{1}{\lambda} \times \frac{du}{u}$$

$$\iff du = -\lambda u \, dx$$

$$= -\lambda e^{-\lambda x} dx$$

$$E[f(X)] = \int_{+\infty}^{0} f(x) \times -\lambda e^{-\lambda x} dx$$

$$= \int_{0}^{+\infty} f(x) \lambda e^{-\lambda x} dx$$

$$= \int_{\mathbb{R}} f(x) \boxed{\lambda e^{-\lambda x} \mathbb{1}_{[0;+\infty[}(x)} dx$$

densité de X

Donc $X \sim \mathcal{E}(\lambda)$.