Feuille 4

exercice 1 :

$f : \mathbb{R}^n \to \mathbb{R}$

$$d_a f : h \longmapsto \nabla_a f \times h$$

$(a_1, \dots, a_n)$ $\qquad$ $\mathbb{R}^n$

$$\nabla_a f \left(\frac{\partial f}{\partial x_1}(a), \dots, \frac{\partial f}{\partial x_n}(a) \right)$$

- $\dfrac{\partial f}{\partial x}(x_0, y_0) = y_0$

- $\dfrac{\partial f}{\partial y}(x_0, y_0) = x_0$

$$d_{\binom{x_0}{y_0}} f : (h_1, h_2) \longmapsto \underbrace{\nabla_{\binom{x_0}{y_0}} f \times \begin{pmatrix} h_1 \\ h_2 \end{pmatrix}}$$

$$\hookrightarrow \begin{pmatrix} y_0 & x_0 \end{pmatrix} \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} := y_0 h_1 + x_0 h_2$$

donc : $\boxed{d_{\binom{x_0}{y_0}} f : (h_1, h_2) \longmapsto y_0 h_1 + x_0 h_2}$

- $g(x,y) = \Big(\underbrace{g_1(x,y)}_{x+y} \; ; \; \underbrace{g_2(x,y)}_{xy} \Big)$

$$J_{\binom{x_0}{y_0}} g \qquad \begin{pmatrix} \dfrac{\partial g_1}{\partial x}(x_0,y_0) & \dfrac{\partial g_1}{\partial y}(x_0,y_0) \\[2ex] \dfrac{\partial g_2}{\partial x}(x_0,y_0) & \dfrac{\partial g_2}{\partial x}(x_0,y_0) \end{pmatrix}$$

$$f : \mathbb{R}^n \longrightarrow \mathbb{R}^p$$
$$(x_1, \dots, x_n) \longmapsto (f_1, \dots, f_p)$$

$$J_a f \quad \underbrace{\begin{pmatrix} \frac{\partial f_1}{\partial x_1}(a) & \cdots & \frac{\partial f_1}{\partial x_n}(a) \\ \vdots & & \vdots \\ \frac{\partial f_p}{\partial x_1}(a) & \cdots & \frac{\partial f_p}{\partial x_n}(a) \end{pmatrix}}_{\text{taille } p \times n}$$

$$J_{\binom{x_0}{y_0}} g \quad \begin{pmatrix} 1 & 1 \\ y_0 & x_0 \end{pmatrix}$$

$$d_{\binom{x_0}{y_0}} g : (h_1, h_2) \longmapsto \underbrace{J_{\binom{x_0}{y_0}} g \times \begin{pmatrix} h_1 \\ h_2 \end{pmatrix}}$$

$$\begin{pmatrix} 1 & 1 \\ y_0 & x_0 \end{pmatrix} \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} = \begin{pmatrix} h_1 + h_2 \\ y_0 h_1 + x_0 h_2 \end{pmatrix}$$

donc :

$$\boxed{d_{\binom{x_0}{y_0}} g : (h_1, h_2) \longmapsto (h_1 + h_2 \ ; \ y_0 h_1 + x_0 h_2).}$$

$$\boxed{\begin{array}{l} d_a(f\circ g) = d_{g(a)} f \circ d_a g \\ \text{E,} \\ \qquad J_a(f\circ g) = J_{g(a)} f \times J_a g \end{array}}$$

$$d_{\binom{x_0}{y_0}}(f\circ g) = d_{\underbrace{g(x_0,y_0)}_{(g_1(x_0,y_0)\,;\,g_2(x_0,y_0))}} f \circ d_{\binom{x_0}{y_0}} g$$

or $\quad d_{\binom{x_0}{y_0}} f(h_1, h_2) = h_1 y_0 + h_2 x_0$

d'où $\quad d_{g(x_0,y_0)} f(h_1,h_2) = h_1\, g_2(x_0,y_0) + h_2\, g_1(x_0,y_0)$

$$= h_1 x_0 y_0 + h_2 (x_0 + y_0)$$

on va prendre $\begin{cases} h_1 = h_1 + h_2 \\ h_2 = y_0 h_1 + x_0 h_2 \end{cases}$

Donc :

$$d_{\binom{x_0}{y_0}}(f \circ g)(h_1, h_2)$$

$$= (h_1 + h_2) x_0 y_0 + (y_0 h_1 + x_0 h_2) \times (x_0 + y_0)$$

$$= h_1 x_0 y_0 + h_2 x_0 y_0 + x_0 y_0 h_1 + y_0^2 h_1 + x_0^2 h_2 + x_0 y_0 h_2$$

$$= x_0^2 h_2 + y_0^2 h_1 + 2 x_0 y_0 h_1 + 2 x_0 y_0 h_2$$

$$(f \circ g)(x, y) = f(x+y, xy)$$

$$= (x+y) xy = x^2 y + x y^2$$

exercice 3 :

1) $g_u(t) = f(x_0 + tu)$

Comme f est différentiable au point x_0, alors il existe une application linéaire $d_{x_0}f$ telle que, pour tout $h \in \mathbb{R}^n$.

$$f(x_0+h) = f(x_0) + d_{x_0}f(h) + \underset{h\to 0}{o}(\|h\|)$$

on prend $h = tu \in \mathbb{R}^n$.

$$f(x_0+tu) = f(x_0) + d_{x_0}f(tu) + \underset{t\to 0}{o}(\|tu\|)$$

$$g_u(t) = f(x_0) + t\,d_{x_0}f(u) + \underset{t\to 0}{o}(|t|\|u\|)$$

$$\boxed{g_u(t) = g_u(0) + t\,d_{x_0}f(u) + \underset{t\to 0}{o}(|t|)}$$

d'où
$$\frac{g_u(t) - g_u(0)}{t - 0}$$
$$= d_{x_0} f(u) + o(1)$$

donc $\lim_{t \to 0} \frac{g_u(t) - g_u(0)}{t - 0} = d_{x_0} f(u)$

Ainsi g est dérivable en $t = 0$ et $\boxed{g_u'(0) = d_{x_0} f(u)}$

2) $g_u(t) = f(tu)$
$$= f(tu_1, tu_2)$$
$$= \frac{(tu_1)^2 \times tu_2}{(tu_1)^2 + (tu_2)^2}$$

$$= \frac{t^3 u_1^2 u_2}{t^2 u_1^2 + t^2 u_2^2}$$

$$= \frac{t^2 (t u_1^2 u_2)}{t^2 (u_1^2 + u_2^2)}$$

$$= t \times \frac{u_1^2 u_2}{u_1^2 + u_2^2} \quad , \quad (u_1, u_2) \neq (0,0)$$

Donc $g_u'(t) = \dfrac{u_1^2 u_2}{u_1^2 + u_2^2} \quad , \quad u \neq 0$

et $\boxed{g_u'(0) = \dfrac{u_1^2 u_2}{u_2^2 + u_2^2} \quad , \quad u \neq 0}$

si $u = 0$, $g_0(t) = f(0,0) = 0$

ainsi $\boxed{g'_u(0) = 0}$

- Si f est différentiable en $(0,0)$, alors d'après la question 1 on aurait que :

$$g'_u(0) = d_{(0,0)}f(u)$$

et $u \longmapsto g'_u(0)$ doit être linéaire.

$$g'_{(1,0)}(0) = 0$$

$$g'_{(0,1)}(0) = 0$$

$(1,0)+(0,1)$

or $g'_{(1,1)}(0) = 1 \neq 0 + 0$

donc $g'_u(0)$ n'est pas linéaire et donc f n'est pas différentiable en $(0,0)$.