exercice 47 :

1) Soit f une fonction mesurable et positive.

$$E[f(u,v)]$$

$$= \int_{(\mathbb{R}_+)^2} f\left((xy)^{\frac{1}{4}}, \left(\frac{x}{y}\right)^{\frac{1}{4}}\right) \underbrace{\frac{e^{-(xy)^{\frac{1}{4}}}}{4\pi(y\sqrt{x}+x\sqrt{y})}}_{f_{X,Y}(x,y)} dx\,dy$$

on pose $\begin{cases} u = (xy)^{\frac{1}{4}} := f(x,y) \\ v = \left(\frac{x}{y}\right)^{\frac{1}{4}} := g(x,y) \end{cases}$

$$u^2 = (xy)^{\frac{2}{4}} \qquad v^2 = \left(\frac{x}{y}\right)^{\frac{2}{4}}$$

$$\begin{cases} x = u^2 v^2 \\ y = \dfrac{u^2}{v^2} \end{cases}$$

$$J_{(x,y)\to(u,v)} = \begin{pmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{pmatrix}$$

$$= \begin{pmatrix} 2uv^2 & 2vu^2 \\ \dfrac{2u}{v^2} & -\dfrac{2u^2}{v^3} \end{pmatrix}$$

$$|\det J| = \left| -\frac{4u^3v^2}{v^3} - \frac{4u^3v}{v^2} \right|$$

$$= \left| -\frac{4u^3}{v} - \frac{4u^3}{v} \right|$$

$$= \frac{8u^3}{v}$$

donc $\quad dx\,dy = \dfrac{8u^3}{v}\,du\,dv$

$$y\sqrt{x} = \frac{u^2}{v^2} \times uv = \frac{u^3}{v}$$

$$x\sqrt{y} = u^2v^2 \times \frac{u}{v} = u^3v$$

$$E\left[g(u,v)\right]$$

$$= \int_0^{+\infty}\int_0^{+\infty} g(u,v)\,\frac{e^{-u}}{4\pi\left(\frac{u^3}{v}+u^3v\right)} \times \frac{8u^3}{v}\,du.dv$$

$$= \int_0^{+\infty}\int_0^{+\infty} \frac{g(u,v)\,e^{-u}}{\frac{4\pi\cancel{u^3}}{\cancel{v}}\left(1+v^2\right)} \times \frac{8\cancel{u^3}}{\cancel{v}}\,dudv$$

$$= \int_0^{+\infty}\int_0^{+\infty} g(u,v)\,\frac{2e^{-u}}{\pi\left(1+v^2\right)}\,dudv$$

$$= \int_{\mathbb{R}^2} f(u,v) \times \underbrace{\frac{2e^{-u}}{\pi(1+v^2)} \mathbb{1}_{u\geq 0} \mathbb{1}_{v \geq 0}}_{f_{U,V}(u,v)} \, du\,dv$$

donc la loi du couple (U,V) est :

$$f_{U,V}(u,v) = \frac{2e^{-u}}{\pi(1+v^2)} \mathbb{1}_{u\geq 0} \mathbb{1}_{v\geq 0}$$

2) $$f_{U,V}(u,v) = \underbrace{\left(e^{-u} \mathbb{1}_{u\geq 0}\right)}_{f_U(u)} \times \underbrace{\left(\frac{2 \mathbb{1}_{v\geq 0}}{\pi(1+v^2)}\right)}_{f_V(v)}$$

on a :

$$f_{\mathcal{U},V}(u,v) = f_{\mathcal{U}}(u) \times f_V(v)$$

donc $\mathcal{U}$ et V sont indépendantes.

3) De plus,

$$f_{\mathcal{U}}(u) = e^{-u} \mathbb{1}_{u>0} \sim \mathcal{E}(1)$$

$$f_V(v) = \frac{2 \, \mathbb{1}_{v>0}}{\pi(1+v^2)}.$$

(x, y) : anciennes variables

(u, v) : nouvelles variables.

- $J_{(x,y) \longrightarrow (u,v)} = \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix}$

$\Rightarrow \quad dx\,dy = |\det J| \, du\,dv$

- $J_{(u,v) \longrightarrow (x,y)} = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix}$

$\Rightarrow \quad dx\,dy = \dfrac{1}{|\det J|} \, du\,dv$