Worksheet 4

exercice 2 :

on pose $x = (x_1, \dots, x_n) \in \mathbb{R}^n$

$y = (1, \dots, 1) \in \mathbb{R}^n$

d'après Cauchy-Schwarz

$$(x \cdot y)^2 \leq \|x\|^2 \times \|y\|^2$$

$$(x_1 + \dots + x_n)^2 \leq (x_1^2 + \dots + x_n^2)(\underbrace{1 + \dots + 1}_{n})$$

$$(\underbrace{x_1 + \dots + x_n}_{n})^2 \leq (\underbrace{x_1^2 + \dots + x_n^2}_{n}) \times n$$

$$n^2 \leq n \times n = n^2$$

donc on a un cas d'égalité de Cauchy-Schwarz :

Donc $x = \lambda y$, $\lambda \in \mathbb{R}$

d'où $(x_1, \dots, x_n) = \lambda(1, \dots, 1)$

$$(x_1, \dots, x_n) = (\lambda, \dots, \lambda)$$

donc $\boxed{x_1 = \dots = x_n = \lambda.}$

et alors on a :

$$\lambda + \dots + \lambda = n$$

$\iff n \times \lambda = n \implies \lambda = 1$

donc $\boxed{x_1 = \cdots = x_n = 1.}$

Worsheet 5

exercice 1 :

1) $\cdot$ $y' - y = 0 \quad (E_h)$

$y' = 1 \times y$

$\Rightarrow y(x) = C e^{x}, \; C \in \mathbb{R}.$

$\cdot$ on pose $y_p(x) = C(x) e^{x}$

$$y_p'(x) = C'(x)e^x + C(x)e^x$$

$$y_p' - y_p = e^{-x^2}$$

$$\Leftrightarrow C'(x)e^x + \cancel{C(x)e^x} - \cancel{C(x)e^x} = e^{-x^2}$$

$$\Rightarrow \boxed{C'(x)e^x = e^{-x^2}}$$

$$\Leftrightarrow C'(x) = e^{-x^2-x}$$

$$\Rightarrow C(x) = \underbrace{\int_0^x e^{-t^2-t}\,dt}_{u(x)}$$

donc $\boxed{y_p(x) = u(x)e^x}$

donc les solutions sont :

$$y(x) = Ce^{x} + u(x)e^{x}, \quad \underline{\underline{C \in \mathbb{R}}}$$

$$= e^{x}(C + u(x)), \quad C \in \mathbb{R}$$

2) $y(x) = e^{x}(C + u(x))$

$$\lim_{x \to +\infty} u(x) = \int_0^{+\infty} \underbrace{e^{-t^2-t}}_{\leq e^{-t^2}} dt$$

$$< +\infty$$

et $\lim_{x \to +\infty} e^{x} = +\infty$

on pose $C = -\int_0^{+\infty} e^{-t^2-t}\,dt$

$$C + u(x) = -\int_0^{+\infty} e^{-t^2-t}\,dt + \int_0^{x} e^{-t^2-t}\,dt$$

$$= \int_{+\infty}^{0} e^{-t^2-t}\,dt + \int_0^{x} e^{-t^2-t}\,dt$$

$$= \int_{+\infty}^{x} e^{-t^2-t}\,dt$$

$$g(x) = -e^{x}\int_x^{+\infty} e^{-t^2-t}\,dt$$

$$|g(x)| \leq e^{x}\int_x^{+\infty} e^{-t^2-t}\,dt$$

$$\leq e^{x} \times e^{-x^2-x} \times \ldots$$

$$\leq e^{-x^2} \xrightarrow[x \to +\infty]{} 0$$

donc $y(x) \xrightarrow[x \to +\infty]{} 0$

<u>exercice 10</u> :

$$\begin{cases} x' = x + y \\ y' = -x + 2y + z \\ z' = x + z \end{cases}$$

on pose $X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$

et $X' = AX$

$$\begin{pmatrix} 1 & 1 & 0 \\ -1 & 2 & 1 \\ 1 & 0 & 1 \end{pmatrix}$$

$$\chi_A(\lambda) = \det \begin{pmatrix} 1-\lambda & 1 & 0 \\ -1 & 2-\lambda & 1 \\ 1 & 0 & 1-\lambda \end{pmatrix}$$

$$= - \begin{vmatrix} 1-\lambda & 1 \\ 1 & 0 \end{vmatrix} + (1-\lambda) \times \begin{vmatrix} 1-\lambda & 1 \\ -1 & 2-\lambda \end{vmatrix}$$

$$= 1 + (1-\lambda)\left[(1-\lambda)(2-\lambda) + 1\right]$$

$$= -\lambda^3 + 4\lambda^2 - 6\lambda + 4$$

$$= (2-\lambda)\left(\underbrace{(1-\lambda)^2 + 1}\right)$$

$$(1-\lambda)^2 = -1 \Leftrightarrow \begin{cases} 1-\lambda = i \\ 1-\lambda = -i \end{cases}$$

on a : $\begin{cases} \lambda_1 = 2 \\ \lambda_2 = 1+i \\ \lambda_3 = 1-i \end{cases}$

$$e_2 = (1, 1, 1)$$

$$e_{1+i} = (i, -1, 1)$$

$$e_{1-i} = (-i, -1, 1) = \overline{e_{1+i}}$$

d'aprés ton cours tu peux directement écrire :

$$x(t) = k_1 e^{2t} v_2 +$$

$\alpha = Re(1 \pm i)$

$$e^{t} \left(k_2 Re\left(e^{it} v_{1+i}\right) + k_3 Im\left(e^{it} v_{1+i}\right) \right)$$

$$= k_1 e^{2t} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + k_2 e^{t} \begin{pmatrix} -\sin t \\ -\cos t \\ \cos t \end{pmatrix}$$

$$+ k_3 e^{t} \begin{pmatrix} \cos t \\ -\sin t \\ \sin t \end{pmatrix}.$$