TD 6

exercice 6 :

1) $u_0 = 1$

$u_1 = 1$

$u_2 = 2$

$u_3 = 3$

$u_4 = 5$

$u_5 = 8$

$u_6 = 13$

$u_7 = 21$

$u_8 = 34$

$u_9 = 55$

2) $A = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$

$$P_A(\lambda) = \begin{vmatrix} -\lambda & 1 \\ 1 & 1-\lambda \end{vmatrix}$$

$$= -\lambda(1-\lambda) - 1$$

$$= -\lambda + \lambda^2 - 1$$

$$= \lambda^2 - \lambda - 1$$

$$= 5$$

$$\lambda_1 = \frac{1-\sqrt{5}}{2}$$

$$\lambda_2 = \frac{1+\sqrt{5}}{2}$$

donc $P_A(\lambda) = (\lambda - \lambda_1)(\lambda - \lambda_2)$

donc il est scindé à racines simples.

Donc A est diagonalisable.

3) $E_{\lambda_1} = \ker(A - \lambda_1 I_2)$

$$= \ker \begin{pmatrix} \frac{-1+\sqrt{5}}{2} & 1 \\ 1 & 1 - \left(\frac{1-\sqrt{5}}{2}\right) \end{pmatrix}$$

$$= \ker \begin{pmatrix} \frac{-1+\sqrt{5}}{2} & 1 \\ 1 & \frac{1+\sqrt{5}}{2} \end{pmatrix}$$

$$\Longleftrightarrow \begin{cases} \frac{-1+\sqrt{5}}{2}x + y = 0 \\ x + \frac{1+\sqrt{5}}{2}y = 0 \end{cases}$$

$$\Longleftrightarrow \begin{cases} y = \frac{1-\sqrt{5}}{2}x \\ x = \frac{-(1+\sqrt{5})}{2}y \end{cases}$$

donc $E_{\lambda_1} = \mathrm{Vect}\left\{\left(1 ; \frac{1-\sqrt{5}}{2}\right)\right\}$

$$E_{\lambda_2} = \ker(A - \lambda_2 I_2)$$

$$= \ker \begin{pmatrix} \frac{-1-\sqrt{5}}{2} & 1 \\ 1 & 1 - \frac{(1+\sqrt{5})}{2} \end{pmatrix}$$

$$= \ker \begin{pmatrix} -\frac{1-\sqrt{5}}{2} & 1 \\ 1 & \frac{1-\sqrt{5}}{2} \end{pmatrix}$$

$$x = \frac{\sqrt{5}-1}{2} y$$

donc $E_{\lambda_2} = \text{Vect}\left\{\left(\frac{\sqrt{5}-1}{2} ; 1\right)\right\}$

on pose :

$$D = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

$$P = \begin{pmatrix} 1 & \frac{\sqrt{5}-1}{2} \\ \frac{1-\sqrt{5}}{2} & 1 \end{pmatrix}$$

telle que $D = P^{-1} A P$.

4) $A = PDP^{-1}$

$$A^n = PD^nP^{-1}$$

$$D^n = \begin{pmatrix} \lambda_1^n & 0 \\ 0 & \lambda_2^n \end{pmatrix}$$

$$= \lambda_1^n \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \lambda_2^n \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

$$A^n = P\left[\lambda_1^n \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \lambda_2^n \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right]P^{-1}$$

$$= \lambda_1^n \underbrace{P\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}P^{-1}}_{A_1} + \lambda_2^n \underbrace{P\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}P^{-1}}_{A_2}$$

$$P = \begin{pmatrix} 1 & \frac{\sqrt{5}-1}{2} \\ \frac{1-\sqrt{5}}{2} & 1 \end{pmatrix}$$

$$P^{-1} = \frac{1}{-1} \begin{pmatrix} 1 & \frac{1-\sqrt{5}}{2} \\ \frac{\sqrt{5}-1}{2} & 1 \end{pmatrix}$$

$$= \begin{pmatrix} -1 & \frac{\sqrt{5}-1}{2} \\ \frac{1-\sqrt{5}}{2} & -1 \end{pmatrix}$$

$$\boxed{\begin{aligned} & U_n = \begin{pmatrix} u_n \\ v_n \end{pmatrix} \\ \Rightarrow\ & U_{n+1} = A U_n \end{aligned}}$$

on sait que :

$$\boxed{U_n = A^n U_0}$$

avec $U_0 = \begin{pmatrix} u_0 \\ v_0 \end{pmatrix}$

5) $U_{n+1} = A U_n$

alors on a : $U_n = A^n U_0$

avec $U_0 = \begin{pmatrix} u_0 \\ v_0 \end{pmatrix} = \begin{pmatrix} u_0 \\ u_1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

et $A^n = \lambda_1^n A_1 + \lambda_2^n A_2$

donc :

$$\mathcal{U}_n = \left(\lambda_1^n A_1 + \lambda_2^n A_2\right)\mathcal{U}_0$$

$$\begin{pmatrix} u_n \\ v_n \end{pmatrix} = \lambda_1^n A_1 \mathcal{U}_0 + \lambda_2^n A_2 \mathcal{U}_0$$

$$= \lambda_1^n \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \lambda_2^n \begin{pmatrix} -1 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$= \lambda_1^n \begin{pmatrix} 3 \\ 1 \end{pmatrix} + \lambda_2^n \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} \overset{c_1}{3}\lambda_1^n \quad \overset{c_2 = 0}{} \\ \lambda_1^n + \lambda_2^n \end{pmatrix} \begin{matrix} \longrightarrow u_n \\ \longrightarrow v_n \end{matrix}$$