Partie A

exercice A. 1 :

1) • $x(t) = (x^0 - t^2 y^0)e^{2t}$

$\hookrightarrow x'(t) = -2ty^0 e^{2t} + (x^0 - t^2 y^0) \times 2e^{2t}$

$= 2\left[(x^0 - t^2 y^0)e^{2t} - ty^0 e^{2t}\right]$

or $2(x(t) - ty(t))$

$= 2((x^0 - t^2 y^0)e^{2t} - ty^0 e^{2t})$

donc $x'(t) = 2(x(t) - ty(t))$

$\cdot\ y(t) = y^0 e^{2t}$

$\hookrightarrow\ y'(t) = 2y^0 e^{2t} = 2y(t)$

Donc $\begin{cases} x' = 2(x - ty) \\ y' = 2y \end{cases}$

et S est solution.

$\cdot\ S(0) = \begin{pmatrix} (x^0 - 0^2 y^0)e^0 \\ y^0 e^0 \end{pmatrix} = \begin{pmatrix} x^0 \\ y^0 \end{pmatrix}$

2) a) $\begin{pmatrix} x^{k+1} \\ y^{k+1} \end{pmatrix} = \begin{pmatrix} x^k \\ y^k \end{pmatrix} + h \begin{pmatrix} 2(x^k - t_k y^k) \\ 2y^k \end{pmatrix}$

b) $\begin{cases} x^{k+1} = x^k + 2h(x^k - t_k y^k) \\ y^{k+1} = y^k + 2h\, y^k \end{cases}$

$$\begin{cases} x^{k+1} = x^k(1+2h) - 2h\,t_k\,y^k \\ y^{k+1} = y^k(1+2h) \end{cases}$$

Ainsi : $$y^{k+1} = y^k \times (1+2h)$$

$$\hookrightarrow \boxed{y^k = y^0 (1+2h)^k}.$$

et on a :

$$x^{k+1} = x^k(1+2h) - 2h\,t_k\,y^0(1+2h)^k$$

on pose :

$$u_k = \frac{x^k}{(1+2h)^k}$$

$$u_{k+1} = \frac{x^{k+1}}{(1+2h)^{k+1}}$$

$$= u_k - \frac{2h t_k y^0}{(1+2h)}$$

$$= u_k - k \times \frac{2h^2 y^0}{(1+2h)}$$

$$\Rightarrow u_k = u_0 - \sum_{i=0}^{k-1} i \times \frac{2h^2 y^0}{(1+2h)}$$

$$= x^0 - \frac{2h^2 y^0}{(1+2h)} \times \sum_{i=0}^{k-1} i$$

$$= x^0 - \frac{\cancel{2}h^2 y^0}{1+2h} \times \frac{(k-1)\times k}{\cancel{2}}$$

$$= x^0 - \frac{h^2 k(k-1) y^0}{1+2h}$$

donc :

$$\boxed{x^k = (1+2h)^k\left(x^0 - \frac{h^2 k(k-1)\,x^0}{1+2h}\right)}$$

c) $\lim\limits_{h \to 0} x^k(t) = x(t)$

$\lim\limits_{h \to 0} y^k(t) = y(t).$

Partie B

exercice B.2 :

2) $y^{(6)} - 3y^{(4)} + 2y^{(3)} = 0$

$\Rightarrow r^6 - 3r^4 + 2r^3 = 0$

$r^3(r^3 - 3r + 2) = 0$

$r^3(r-1)(r^2+r-2) = 0$

$\Delta = 9$

$r_1 = -2 \qquad r_2 = 1$

$$\boxed{r^3(r-1)^2(r+2) = 0}$$

$$y(t) = C_1 e^{-2t} + (C_2 + C_3 t)e^{t}$$
$$+ (C_4 + C_5 t + C_6 t^2)$$

avec $C_1, C_2, C_3, C_4, C_5, C_6 \in \mathbb{R}$.

exercice B.3 :

1) $(t+1)y' - ty = -1$

$$\Leftrightarrow \quad y' - \frac{t}{t+1} y = -\frac{1}{t+1}$$

• $y' - \dfrac{t}{t+1} y = 0$

on a : $y(t) = Ce^{\int \frac{t}{t+1} dt}$, $C \in \mathbb{R}$

$$\int \frac{t}{t+1}\,dt$$

$$= \int 1 - \frac{1}{t+1}\,dt$$

$$= \int 1 - \int \frac{1}{t+1}$$

$$\Rightarrow t - \ln(t+1)$$

$$y(t) = C\,e^{t-\ln(t+1)}, \quad C \in \mathbb{R}$$

$$= C\,e^{t} \times e^{\ln\left(\frac{1}{t+1}\right)}, \quad C \in \mathbb{R}$$

$$= C\,\frac{e^{t}}{t+1}, \quad C \in \mathbb{R}.$$

• $y(t) = C(t) \times \dfrac{e^t}{1+t}$

$$\Rightarrow C'(t) \times \frac{e^t}{1\cancel{+t}} = \frac{-1}{\cancel{1+t}}$$

$$C'(t) = -1 \times e^{-t} = -e^{-t}$$

$$C(t) = e^{-t}$$

donc $y(t) = e^{-t} \times \dfrac{e^t}{1+t} = \dfrac{1}{1+t}$

Donc :

$$\boxed{y(t) = C\frac{e^t}{1+t} + \frac{1}{1+t}, \quad C \in \mathbb{R}}$$

$$y(0) = C + 1 = 2$$

$$\Rightarrow C = 2 - 1 = 1$$

Donc :

$$\boxed{y(t) = \frac{1}{1+t}\left(e^t + 1\right)}.$$

$\lim_{t \to -1} y(t) = +\infty$, donc non prolongeable.

2) • $y'' - 3y' - 4y = 0$

$r^2 - 3r^2 - 4 = 0$

$\Delta = 25$

$r_1 = -1$ et $r_2 = 4$

doc :

$y(t) = C_1 e^{-t} + C_2 e^{4t}, \quad C_1, C_2 \in \mathbb{R}$

• $y'' - 3y' - 4y = \underbrace{t}_{at+b}$

on pose $y(r) = at + b$.

$$y''(t) = 0$$

$$y'(t) = a$$

$$y'' - 3y' - 4y$$

$$= -3a - 4at - 4b$$

$$\Rightarrow (-3a - 4b) - 4at = t$$

$$\Leftrightarrow \begin{cases} -4a = 1 \\ -3a - 4b = 0 \end{cases} \Leftrightarrow a = -\frac{1}{4}$$

$$\Leftrightarrow b = \frac{3}{16}$$

$$\boxed{y(t) = -\frac{1}{4}t + \frac{3}{16}}$$

• $y'' - 3y' - 4y = e^{-t}$

• -1 est racine de $r^2 - 3r - 4$

on cherche $y(t) = Ate^{-t}$

$y'(t) = Ae^{-t} - Ate^{-t}$

$y''(t) = -Ae^{-t} - Ae^{-t} + Ate^{-t}$

$$y'' - 3y' - 4y = -2Ae^{-t} + Ate^{-t}$$
$$-3Ae^{-t} + 3Ate^{-t}$$
$$-4Ate^{-t}$$

$$= -5Ae^{-t} + 0Ate^{-t}$$

$$= -5Ae^{-t}$$

$$-5Ae^{-t} = e^{-t}$$

$$\Leftrightarrow -5A = 1$$

$$\Leftrightarrow A = -\frac{1}{5}.$$

Donc : $\boxed{y(t) = -\frac{1}{5}t\,e^{-t}}$

Par superposition, on a une solution particulière

$$y(t) = -\frac{1}{4}t + \frac{3}{16} - \frac{1}{5}t\,e^{-t}$$

Donc :

$$\boxed{y(t) = C_1 e^{-t} + C_2 e^{4t} - \frac{1}{4}t + \frac{3}{16} - \frac{1}{5}t\,e^{-t} \quad \text{avec } C_1, C_2 \in \mathbb{R}}$$