

Calculus Différentielle

exercice 1:

$$f: \mathbb{R}^n \rightarrow \underline{\mathbb{R}}$$

$$d_a f: h \mapsto \nabla_a f \times h$$

$$\cdot \frac{\partial f}{\partial x}(x_0, y_0) = y_0$$

$$\cdot \frac{\partial f}{\partial y}(x, y) = x_0$$

$$d_{(x_0, y_0)} f(h_1, h_2) = \left(\underset{\downarrow \frac{\partial f}{\partial x}}{y_0} \underset{\downarrow \frac{\partial f}{\partial y}}{x_0} \right) \begin{pmatrix} h_1 \\ h_2 \end{pmatrix}$$

$$= y_0 h_1 + x_0 h_2$$

$$f(x_1, x_2, \dots, x_n)$$

$$\hookrightarrow \nabla f = (\partial_{x_1} f \quad \partial_{x_2} f \quad \dots \quad \partial_{x_n} f)$$

$$\cdot \frac{\partial f}{\partial x}(x_0, y_0) = \left(\overset{\partial_x(x+y)}{\downarrow} 1 \ ; \ \overset{\partial_x(x+y)}{\downarrow} y_0 \right)$$

$$\cdot \frac{\partial f}{\partial y}(x_0, y_0) = \left(1 \ ; \ x_0 \right)$$

$$\Rightarrow J_{(x_0, y_0)} f = \begin{pmatrix} 1 & 1 \\ y_0 & x_0 \end{pmatrix}$$

$$\overbrace{\begin{pmatrix} \partial_x g_1 & \partial_y g_1 \\ \partial_x g_2 & \partial_y g_2 \end{pmatrix}}$$

$$d_a f : h \mapsto J_a f \cdot h$$

$$\hookrightarrow f : \mathbb{R}^n \rightarrow \mathbb{R}^p$$

$$d_{\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}} g(h_1, h_2) = \begin{pmatrix} 1 & 1 \\ y_0 & x_0 \end{pmatrix} \begin{pmatrix} h_1 \\ h_2 \end{pmatrix}$$

$$= (h_1 + h_2, y_0 h_1 + x_0 h_2)$$

$$d_a (f \circ g) = d_{g(a)} f \circ d_a g$$

$$d_{\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}} g(h_1, h_2) = (h_1 + h_2, y_0 h_1 + x_0 h_2)$$

$$g(x_0, y_0) = (x_0 + y_0, x_0 y_0)$$

$$d_{\left(\begin{smallmatrix} x_0 \\ x_0 + y_0 \\ x_0 y_0 \\ y_0 \end{smallmatrix}\right)} f = x_0 y_0 h_1 + (x_0 + y_0) h_2$$

d'au :

$$d_{\left(\begin{smallmatrix} x_0 \\ y_0 \end{smallmatrix}\right)}(f \circ g)(h_1, h_2)$$

$$= x_0 y_0 (h_1 + h_2) + (x_0 + y_0)(y_0 h_1 + x_0 h_2)$$

$$= x_0 y_0 h_1 + x_0 y_0 h_2 + x_0 y_0 h_1 + x_0^2 h_2 + y_0^2 h_1 + y_0 x_0 h_2$$

$$= (y_0^2 + 2x_0 y_0) h_1 + (x_0^2 + 2x_0 y_0) h_2$$

$$\begin{aligned}
 & \cdot f(g(x, y)) \\
 &= f(x+y, xy) \\
 &= (x+y)xy \\
 &= x^2y + xy^2
 \end{aligned}$$

$$\frac{\partial f}{\partial x}(x_0, y_0) = 2x_0y_0 + y_0^2$$

$$\frac{\partial f}{\partial y}(x_0, y_0) = x_0^2 + 2y_0x_0$$

$$\begin{aligned}
 d_{\left(\begin{smallmatrix} x_0 \\ y_0 \end{smallmatrix}\right)}(f \circ g)(h_1, h_2) \\
 = \begin{pmatrix} 2x_0y_0 + y_0^2 & x_0^2 + 2y_0x_0 \end{pmatrix} \begin{pmatrix} h_1 \\ h_2 \end{pmatrix}
 \end{aligned}$$

$$= (2x_0y_0 + y_0^2)h_1 + (x_0^2 + 2y_0x_0)h_2$$

exercice 6:

$$f(x, y) = \begin{cases} (x^2 + y^2) \sin\left(\frac{1}{\sqrt{x^2 + y^2}}\right) & \text{Si } (x, y) \neq (0, 0) \\ 0 & (x, y) = (0, 0) \end{cases}$$

• continuité:

on pose $\begin{cases} x = r \cos(\theta) \\ y = r \sin(\theta) \end{cases}, r > 0$

$$f(r \cos \theta, r \sin \theta)$$

$$= (r^2 \cos^2 \theta + r^2 \sin^2 \theta) \times \sin \left(\frac{1}{\sqrt{r^2 \cos^2 \theta + r^2 \sin^2 \theta}} \right)$$

$$= (r^2 (\cos^2 + \sin^2)) \times \sin \left(\frac{1}{\sqrt{r^2 (\cos^2 + \sin^2)}} \right)$$

$$= r^2 \times \sin \left(\frac{1}{r} \right) \leq 1$$

$$\text{or } |f(r \cos \theta, r \sin \theta)|$$

$$\leq r^2 \times 1$$

$$\xrightarrow{r \rightarrow 0} 0$$

done f est continue en $(0, 0)$.

sq :

$$|f(x, y)|$$

$$= \left| (x^2 + y^2) \sin \left(\frac{f}{\sqrt{x^2 + y^2}} \right) \right|$$

$$\leq x^2 + y^2$$

$$\xrightarrow{(x, y) \rightarrow \underline{(0, 0)}} 0 + 0 = 0 = f(0, 0)$$

ainsi: $\lim_{(x, y) \rightarrow (0, 0)} f(x, y) = f(0, 0)$

et f est continue en $(0, 0)$.

• differentiable :

$$\frac{\partial f}{\partial x}(a,b) = \lim_{h \rightarrow 0} \frac{f(a+h, b) - f(a,b)}{h}$$

$$\frac{\partial f}{\partial y}(a,b) = \lim_{h \rightarrow 0} \frac{f(a, b+h) - f(a,b)}{h}$$

$$\cdot \lim_{h \rightarrow 0} \frac{f(0+h, 0) - f(0,0)}{h}$$

$$= \lim_{h \rightarrow 0} \frac{h^2 \sin\left(\frac{1}{|h|}\right)}{h}$$

$$= \lim_{h \rightarrow 0} \underbrace{h \sin\left(\frac{1}{|h|}\right)}_{\leq h}$$

$$= 0$$

donc $\frac{\partial f}{\partial x}(0, c) = 0$

de même, $\lim_{h \rightarrow 0} \frac{f(0, c+h) - f(0, c)}{h}$
 $= 0 \dots = 0$

donc $\frac{\partial f}{\partial y}(c, c) = 0$

f est différentiable en a
 si il existe une application
 linéaire df telle que :

$$f(a+h) = f(a) + df(h) + o(\|h\|)$$

ou $\frac{\|f(a+h) - f(a) - df(h)\|}{\|h\|} \xrightarrow{h \rightarrow 0} 0$

sq: En pratique on montre
que f est $\mathcal{C}^1$.

$\mathcal{C}^1 \Rightarrow$ dérivées partielles
existent + continues.

La seule différentielle en $(0,0)$
possible est nulle.

on a:

$$\frac{|f(0+h) - f(0,0) - d_0 f(h)|}{\|h\|}$$

$$= \frac{|f(h_1, h_2) - 0 - 0|}{\sqrt{h_1^2 + h_2^2}}$$

$$= \frac{|f(h_1, h_2)|}{\sqrt{h_1^2 + h_2^2}}$$

$$= \frac{|(h_1^2 + h_2^2) \sin\left(\frac{1}{\sqrt{h_1^2 + h_2^2}}\right)|}{\sqrt{h_1^2 + h_2^2}}$$

$$\leq \frac{(h_1^2 + h_2^2)^{\frac{1}{2}}}{\sqrt{h_1^2 + h_2^2}^{\frac{1}{2}}}$$

$$= \sqrt{h_1^2 + h_2^2}$$

$$\xrightarrow{(h_1, h_2) \rightarrow (0,0)} 0$$

donc f est différentiable en $(0,0)$. $\square$

$$\bullet \frac{\partial f}{\partial x}(x, y) = 2x \sin\left(\frac{1}{\sqrt{x^2+y^2}}\right) - \frac{x \cos\left(\frac{1}{\sqrt{x^2+y^2}}\right)}{\sqrt{x^2+y^2}}$$

$$\frac{\partial f}{\partial x}(x, 0) = 2x \sin\left(\frac{1}{x}\right) - \frac{\cancel{x} \cos\left(\frac{1}{x}\right)}{\cancel{x}}$$

$$= \underbrace{2x \sin\left(\frac{1}{x}\right)}_{\leq 2x} - \underbrace{\cos\left(\frac{1}{x}\right)}_X$$

$\frac{x-x_0}{x-x_0} > 0$

n' admet pas de limite quand $x \rightarrow c^+$.

Donc $\frac{\partial f}{\partial x}$ n'est pas continue

en $(0,0)$.

Donc $\int$ n'est pas $\in \mathbb{1}$
en $(0,0)$