

Calcul différentiel

exercice 0 :

$$\bullet f(x, y) = x^2 - y^2$$

$$\frac{\partial f}{\partial x}(x, y) = 2x$$

$$\frac{\partial f}{\partial y}(x, y) = -2y$$

$$\begin{cases} 2x = 0 \\ 2y = 0 \end{cases} \iff \begin{cases} x = 0 \\ y = 0 \end{cases}$$

donc f admet un unique point critique $(0, 0)$.

$$\frac{\partial f}{\partial x^2}(x, y) = 2$$

$$\frac{\partial f}{\partial y^2}(x, y) = -2$$

$$\frac{\partial f}{\partial x \partial y}(x, y) = 0 = \frac{\partial f}{\partial y \partial x}(x, y)$$

$$\text{donc } H_{\begin{pmatrix} x \\ y \end{pmatrix}} f = \begin{pmatrix} \overset{x^2}{2} & \overset{xy}{0} \\ \underset{yx}{0} & \underset{y^2}{-2} \end{pmatrix}$$

$$\text{et } H_{\begin{pmatrix} 0 \\ 0 \end{pmatrix}} f = \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix},$$

$$\det H_{\begin{pmatrix} 0 \\ 0 \end{pmatrix}} f = -4 < 0$$

Donc $(0, 0)$ est un point selle.

$$f: \mathbb{R}^2 \rightarrow \mathbb{R}$$

① trouver les points critiques

$$\nabla f(x, y) = (0, 0)$$

② construire $H(x, y)$

$$\rightarrow \det H(x_0, y_0) < 0$$

$\Rightarrow (x_0, y_0)$ point selle

$$\rightarrow \det H(x_0, y_0) > 0$$

$$* \det H(x_0, y_0) > 0$$

$\hookrightarrow (x_0, y_0)$ min local.

$$* \text{tr } H(x_0, y_0) < 0$$

$L(x_0, y_0)$ max local.

$$g(x, y) = x^2(1+y)^3 + y^2.$$

$$\frac{\partial g}{\partial x}(x, y) = 2x(1+y)^3$$

$$\frac{\partial g}{\partial y}(x, y) = x^2 \times 3 \times \overbrace{(1+y)}^{(1+y)'} \times (1+y)^2 + 2y$$

$$= 3x^2(1+y)^2 + 2y.$$

$$\begin{cases} 2x(1+y)^3 = 0 \\ 3x^2(1+y)^2 + 2y = 0 \end{cases}$$

$$\Rightarrow 2x = 0 \quad \text{ou} \quad (1+y)^3 = 0$$

$$x = 0 \quad \text{ou} \quad y = -1$$

• Si $x = 0$, $2y = 0 \Rightarrow y = 0$
donc $(0, 0)$ point critique

• Si $y = -1$, $2x - 1 = 0$

$$-2 = 0 \quad \text{impossible}$$

pas possible.

Donc g admet un unique point critique $(0, c)$.

$$\frac{\partial g}{\partial x^4}(x, y) = 2(1+y)^3$$

$$\frac{\partial g}{\partial y^4}(x, y) = 3x^2 \times 2 \times 1 \times (1+y) + 2$$

$$= 6x^2(1+y) + 2.$$

$$\frac{\partial g}{\partial x \partial y}(x, y) = 6x(1+y)^2$$

$$= \frac{\partial g}{\partial y \partial x}(x, y)$$

$\hookrightarrow g$ est C^2 + Schwarz

$$H_{\begin{pmatrix} x \\ y \end{pmatrix}} g = \begin{pmatrix} 2(1+y)^3 & 6x(1+y)^2 \\ 6x(1+y)^2 & 6x^2(1+y)+2 \end{pmatrix}$$

$$\text{et } H_{\begin{pmatrix} 0 \\ 0 \end{pmatrix}} g = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$

$$\det H_{\begin{pmatrix} 0 \\ 0 \end{pmatrix}} g = 4 > 0$$

$$\text{et } \text{tr } H_{\begin{pmatrix} 0 \\ 0 \end{pmatrix}} g = 4 > 0$$

donc $(0, 0)$ est un minimum local.

$$\text{or } g(0,0) = 0$$

$\nabla f(0,0)$ serait $\begin{cases} \text{max} \\ \text{min} \end{cases}$ global,
alors :

$$\forall (x,y) \in \mathbb{R}^2, \quad g(x,y) \begin{cases} \leq \\ \geq \end{cases} g(0,0)$$

$$\begin{aligned} g(3, -2) &= 3^2 \times (-1)^3 + 4 \\ &= -9 + 4 \\ &= -5 < 0 \end{aligned}$$

$$\text{donc } g(3, -2) < g(0,0) = 0$$

Ainsi $(0,0)$ n'est pas min
global

$$\text{Def: } g(x, y) = x^4 + y^4$$

$(0, 0)$ seul point critique

$\Rightarrow$ on montre que $(0, 0)$ est
un min local.

$$\text{or } g(0, 0) = 0 \text{ et}$$

$$\forall (x, y) \in \mathbb{R}^2, \quad g(x, y) = x^4 + y^4 \\ \geq 0 = g(0, 0)$$

exercice 5:

$$g(x, y) = -x^2y + \frac{1}{2}y^2 + y.$$

$$\frac{\partial g}{\partial x}(x, y) = -2xy$$

$$\frac{\partial g}{\partial y}(x, y) = -x^2 + y + 1$$

$$\begin{aligned} & \rightarrow xy = \frac{0}{-2} = 0 \\ & \begin{cases} -2(xy) = 0 \\ x^2 + y + 1 = 0 \end{cases} \Leftrightarrow \begin{cases} x=0 \text{ or } y=0 \\ -x^2 + y + 1 = 0 \end{cases} \end{aligned}$$

si $x=0$, alors $y=-1$

si $y=0$, alors $x=\pm 1$

done admet trois points critiques $(0, -1), (-1, 0), (1, 0)$.

$$H_{\begin{pmatrix} x \\ y \end{pmatrix}} g = \begin{pmatrix} -2y & -2x \\ -2x & 1 \end{pmatrix}$$

$$\bullet H_{\begin{pmatrix} 0 \\ -1 \end{pmatrix}} g = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\rightarrow \det H_{\begin{pmatrix} 0 \\ -1 \end{pmatrix}} = 2 > 0$$

$$\rightarrow \text{tr } H_{\begin{pmatrix} 0 \\ -1 \end{pmatrix}} = 3 > 0$$

done $(0, -1)$ est un min local

$$\cdot H\left(-\frac{1}{2}\right) g = \begin{pmatrix} 0 & 2 \\ 2 & 1 \end{pmatrix}$$

$$\rightarrow \det H\left(-\frac{1}{2}\right) = -4 < 0$$

donc $(-1, 0)$ est un point selle

$$\cdot H\left(\frac{1}{2}\right) g = \begin{pmatrix} 0 & -2 \\ -2 & 1 \end{pmatrix}$$

$$\rightarrow \det H\left(\frac{1}{2}\right) = -4 < 0$$

donc $(1, 0)$ est un point selle